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Electromagnetic Field Theory

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Contents

Preface

Hier kommt das Vorwort hin

Chapter 1

Electromagnetic Field

1.1 Maxwell's Equations

Electric and magnetic fields that vary with time are governed by physical laws described by a set of equations known collectivily as Maxwell's equations. James Clerk Maxwell (1831-1879) gave the first formulation of these equations in his famous book *Treatise of Electricity and Magnetism* in 1864, in which he proposed the existence of electromagnetic waves. The first experimentell verification of their existence was done by Heinrich Hertz (1857-1894) 1887 in the "Technische Hochschule" Karlsruhe.

To describe the physical phenomena one defines different vector fields that vary with the three spatial coordinates x,y,z and with time t. We start with the introduction of the

$$\begin{array}{rcll} \vec{E} & = & \vec{E}(x,y,z,t) & - \text{ electric field,} \\ \vec{B} & = & \vec{B}(x,y,z,t) & - \text{ magnetic flux density.} \end{array}$$

The electric field $\vec{E}$ and the magnetic flux density $\vec{B}$ are regarded as fundamental in that they give the force on a charge q moving with a velocity $\vec{v}$.

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

This force is called the Lorentz force.

In addition to the $\vec{E}$ and $\vec{B}$ fields, it is convinient to introduce auxiliary fields:

$$\begin{array}{rcll} \vec{D} & = & \vec{D}(x,y,z,t) & - \text{ electric displacement density} \\ \vec{H} & = & \vec{H}(x,y,z,t) & - \text{ magnetic field} \\ \vec{J}_c & = & \vec{J}_c(x,y,z,t) & - \text{ conduction current density} \end{array}$$

In the case of fields in free space or in vacuum we have very simple relations between this fields.

$$\vec{D} = \varepsilon_0 \vec{E}$$
$$\vec{H} = \frac{1}{\mu_0} \vec{B}$$

with

$$\varepsilon_0 = 8,854 \cdot 10^{-12} \frac{As}{Vm} \quad - \quad \text{permitivity of vacuum}$$
$$\mu_0 = 4\pi \cdot 10^{-7} \frac{Vs}{Am} \quad - \quad \text{permeability of vacuum}$$

Since no charges exist in free space the conduction current density field $\vec{J}_c$ is not existing.

Faraday's law

One of the basic laws of electromagnetic phenomena is Faraday's law, which states that a time varying magnetic field $\vec{B}$ generates a electric field.

$$\oint_C \vec{E} \cdot d\vec{s} = -\frac{\partial}{\partial t} \int_A \vec{B} \cdot d\vec{A} \tag{1.1}$$

To explain equation 1.1 see Figure 1.1. According to Faraday's law the time rate

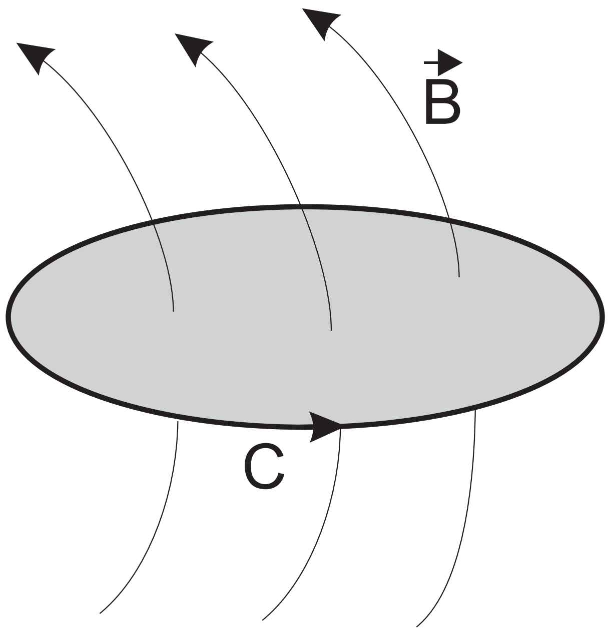

Figure 1.1: Generation of a electric field by a time varying magnetic field

of change of total magnetic flux is equal to the negative value of the total voltage measured around the contour C. It is important to notice that the negative sign is only true if the direction of the magnetic field and the orientation of the contour is right handed as shown in the figure 1.1. With the help of Stoke's theorem, which states that the integral of a vector field around a closed contour is equal to the integral of the normal component of the curl of this vector over any surface having C as its boundary we have

$$\oint_C \vec{E} \cdot d\vec{s} = \int_A (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = -\frac{\partial}{\partial t} \int_A \vec{B} \cdot d\vec{A}$$

hence we find as differential form of Faraday's law.

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1.2}$$

Ampere's law

It formulates how a electric current will built up a magnetic field. A simple example of this is, how a straight wire carrying the total current I is accompanied by concentric magnetic field lines as shown in Figure 1.3 According to the law of

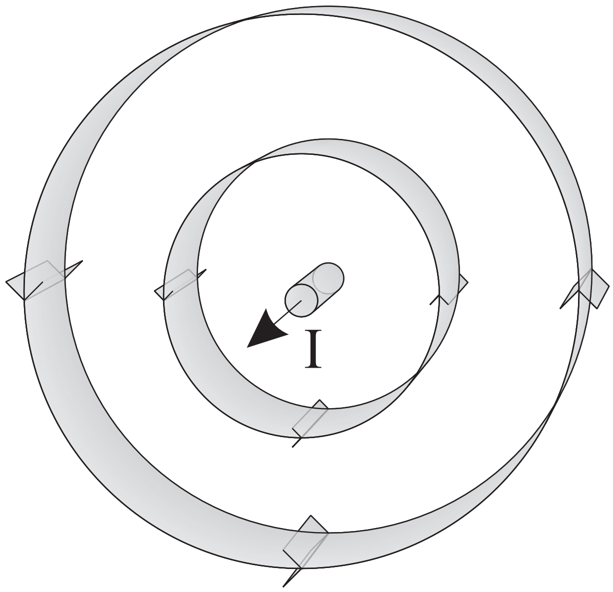

Figure 1.2: Magnetic field due to the current I of a wire

Oerstedt we have for any simple closed contour surrounding the wire

$$\oint_C \vec{H} \cdot d\vec{S} = I$$

It was Maxwell who noticed that not only the conduction current density $\vec{J_c}$ produces a magnetic field but also the so called displacement current density $\vec{J_v} = \partial\vec{D}/\partial t$ has to be considered. So he gave a new formulation of ampere's law.

$$\oint \vec{H} \cdot d\vec{s} = \int_A \left(\frac{\partial\vec{D}}{\partial t} + \vec{J_c}\right) \cdot d\vec{A} \tag{1.3}$$

Again with the help of Stoke's law we find

$$\oint \vec{H} \cdot d\vec{s} = \int_A (\vec{\nabla} \times \vec{H}) d\vec{A}$$

the differential form of ampere's law:

$$\vec{\nabla} \times \vec{H} = \frac{\partial\vec{D}}{\partial t} + \vec{J} \tag{1.4}$$

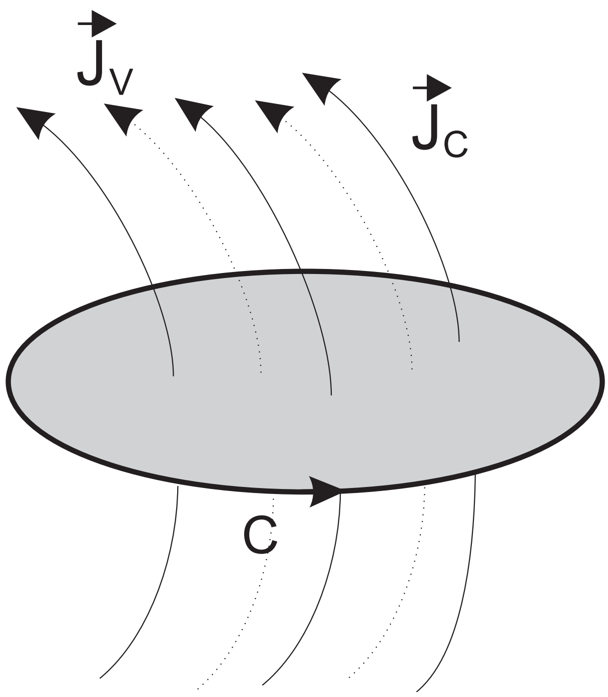

Figure 1.3: Magnetic field due to conduction and displacement current density

Gauss's law

To deduce Gauss's law we will start from equation 1.3 and apply it on a close surface. Since a close surface has no bondary the right handside of equation 1.3

will vansih and we find

$$\frac{\partial}{\partial t}\int_A \vec{D} d\vec{A} = \int_A \vec{J}_c d\vec{A}$$

or integration with respect to time yields

$$\int_A \vec{D} d\vec{A} = \int dt \int_A \vec{J}_c d\vec{A}$$

The left handside of the last equation is of course equal to the total of charges brought into the volume by the flewing currents. Hence we have

$$\oint_A \vec{D}\cdot d\vec{A} = Q$$

To clearify the meaning of this law consider Figure 1.4. It shows an arbitrary vol-

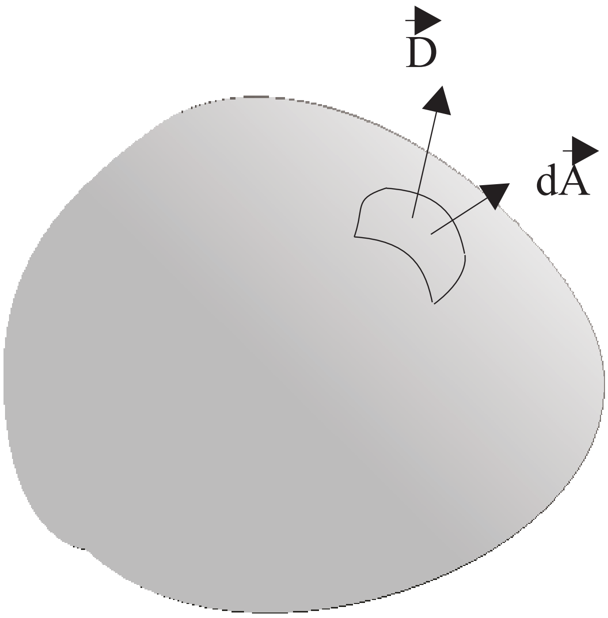

Figure 1.4: Volume with a given charge density ρ

ume bounded by its surface A. Inside this surface we have an charge distribution described by the function of charge density per unit volume ρ. Gauss's law now states that the total electric flux out of the volume V is equal to the net charge contained within V.

$$\oint_A \vec{D}\cdot d\vec{A} = \int_V \rho dV$$

The last equation my be converted to a differential form by using the divergence theorem from Gauss.

$$\oint_A \vec{D}\cdot d\vec{A} = \int_V (\vec{\nabla}\cdot\vec{D}) dV$$

Hence the differential form of Gauss's law reads

$$\vec{\nabla}\cdot\vec{D} = \rho \tag{1.5}$$

In contrast to the electric field there exist not magnetic chrages, so we have in this case

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{1.6}$$

In summary the electromagnetic phenomena is described by the following set of four equations, known as Maxwell's equations.

$$\begin{aligned}
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\
\vec{\nabla}\times\vec{H} &= \frac{\partial\vec{D}}{\partial t} + \vec{J} \\
\vec{\nabla}\cdot\vec{D} &= \rho \\
\vec{\nabla}\cdot\vec{B} &= 0
\end{aligned} \tag{1.7}$$

Phasor representation

Usually we consider only steady state solutions of electromagnetic fields as produced by currents having sinusoidal time dependence. The time dependence of all fields can then be expressed by for example by the real part of the exponential function.

$$\vec{E}(\vec{r},t) =; \vec{E}(\vec{r})\cos(\omega t+\varphi) = \mathbf{Re}\{\vec{E}(\vec{r})\exp(j\varphi)\exp(j\omega t)\}$$

Thus we are ready to define the phasor of a electric field.

$$\underline{\vec{E}}(\vec{r}) = \vec{E}(\vec{r})\exp(j\varphi)$$

Using the phasor representation the derivation with respect to time $\partial/\partial t$ may be replaced by a multiplication with $j\omega$. Hence we get for Maxwell's equation for fields with only sinusoidal time dependence in phasor notation.

$$\begin{aligned}
\vec{\nabla}\times\underline{\vec{E}} &= -j\omega\underline{\vec{B}} \\
\vec{\nabla}\times\underline{\vec{H}} &= j\omega\underline{\vec{D}} + \underline{\vec{J}} \\
\vec{\nabla}\cdot\underline{\vec{D}} &= \underline{\rho} \\
\vec{\nabla}\cdot\underline{\vec{B}} &= 0
\end{aligned} \tag{1.8}$$

1.2 Constitutive relations

In material media the auxiliary fields are defined in terms of the polarization of the material and the fundamental field quantities. The relation between for example $\underline{\vec{D}}$

and $\underline{\vec{E}}$ are known as constitutive relations and must be known before the solution for Maxwell's equations1.7 can be found.

We will first consider the electric case. Figure **??**a shows an undistored atom possessing rotational symmetry. If an electric field $\vec{E}$ is applied to a material body, this force results in a distortion of the atoms or molecules as shown in Figure 1.5b in such a manner as to create effective electric dipoles with a dipole moment $\vec{P}$ per unit volume. With the help of the dipole moment per unit volume we define the

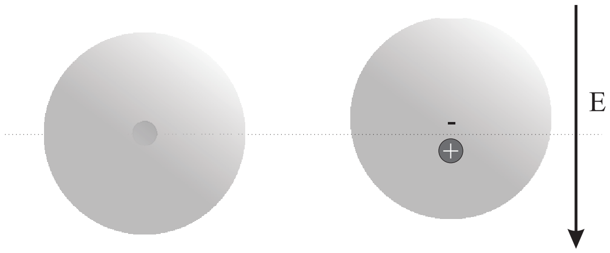

Figure 1.5: Induced dipol by an electric field

displacement density vector $\vec{D}$ by:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

For a great many materials the polarization $\vec{P}$ is in the same direction as the applied electric field $\vec{E}$ and will be proportional to its absolute value. Thus we can rewrite the above equation in the following form

$$\vec{D} = \varepsilon_0 \vec{E} + \chi_e \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

where χ_e is a constant of the material and is called electric susceptibility and ε_r is the relative permittivity of the media. But if we consider sinusoidal time varing fields only for low frequencies the motion of the atoms or molecules will be strict in phase with the applied electric field. For higher frequencies there will be a phase deviation between the electric field and the polarization and this will result in a complex relative permittivity connecting the phasors of the fields

$$\underline{\vec{D}} = \varepsilon_0 \underline{\varepsilon_r} \underline{\vec{E}} = \underline{\varepsilon} \underline{\vec{E}} = (\varepsilon' - j\varepsilon'') \underline{\vec{E}}$$

A complex permittivity $\underline{\varepsilon}$ will allways occur whenever damping effects are present. Loss in a dielectric material may also occur because of a finite conductivity σ. The two mechanisems are indistinguishable as far as external effects related to power dissipation are conserned. For example the curl equation for $\underline{\vec{H}}$ may be written as

$$\vec{\nabla} \times \underline{\vec{H}} = j\omega \underline{\varepsilon} \underline{\vec{E}} + \sigma \underline{\vec{E}}$$

with $\vec{\underline{J}} = \sigma\vec{\underline{E}}$ beeing the conduction current density in the material. Hence we can rewrite the above equation as

$$\vec{\nabla} \times \vec{\underline{H}} \;=\; j\omega[\varepsilon' - j(\varepsilon'' + \frac{\sigma}{\omega})]\vec{\underline{E}}$$

With the help of the last equation we find for the loss tangent of a dielectric medium

$$\tan(\delta_L) \;=\; \frac{\omega\varepsilon'' + \sigma}{\omega\varepsilon'} \;=\; \frac{\varepsilon''}{\varepsilon} + \frac{\sigma}{\omega\varepsilon'}$$

Any measurement of $\tan(\delta_L)$ always includes the effects of finite conductivity σ and damping effects ε''. At microwave frequencies however ε'' is usually much larger than $\sigma/\omega\varepsilon'$ because of the high frequencies. Materials for which $\vec{P}$ is linearly related to $\vec{E}$ and in the same direction as $\vec{E}$ are called linear isotropic materials.

If we consider crystals, these structures lack spherical symmetry. Then the polarization per unit volume will depend on the direction of the applied field. In the generale case were the orientation of the crystal structure has a different orientation with respect to the used coordinate system one gets the following equation between the fields $\vec{\underline{D}}$ and $\vec{\underline{E}}$.

$$\vec{\underline{D}} \;=\; \begin{pmatrix} \underline{D}_x \\ \underline{D}_y \\ \underline{D}_z \end{pmatrix} \;=\; \begin{pmatrix} \underline{\varepsilon}_{xx} & \underline{\varepsilon}_{xy} & \underline{\varepsilon}_{xz} \\ \underline{\varepsilon}_{yx} & \underline{\varepsilon}_{yy} & \underline{\varepsilon}_{yz} \\ \underline{\varepsilon}_{zx} & \underline{\varepsilon}_{zy} & \underline{\varepsilon}_{zz} \end{pmatrix} \cdot \begin{pmatrix} \underline{E}_x \\ \underline{E}_y \\ \underline{E}_z \end{pmatrix}$$

For anisotropic media the permittivity is denoted tensor permittivity.

In the case of the magnetic field in a isotropic linear material we can write an analog equation like in the case of the electric field.

$$\vec{\underline{B}} \;=\; \mu_0(1 + \chi_m)\vec{\underline{H}}$$

were $\mu = \mu_0(1 + \chi_m)$ is called the permeability of the media. As in the electric case, dissipation will cause μ to be a complex frequency dependent parameter with a negative imaginary part.

Also, there are magnetic materials that are isotropic, in particular, ferrites are anisotropic magnetic materials of great importance at microwave frequencies. These exhibit a tensor permeability of the following form

$$[\mu] \;=\; \begin{pmatrix} \mu_1 & j\mu_2 & 0 \\ -j\mu_2 & \mu_1 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$$

when a static magnetic field is applied along the axis for which the permeability is μ_3.

1.3 Static fields

Static fields are defined by the fact that there are no variations in time, hence all derivatives with respect to time vanish and Maxwell's equations 1.7 reduce to

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J} \\ \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \cdot \vec{B} &= 0 \end{aligned} \tag{1.9}$$

1.3.1 Electrostatics

We will start the discussion of these equation with the electric field. Since the static electric field has zero curl the line integral of $\vec{E}$ around an arbitrary closed contour is zero. One calls such a field a conservative field which can always be expressed by the gradient of a scalar function, known as potential function ϕ, since we have $\vec{\nabla} \times (\vec{\nabla}\phi) = 0$. Thus we can express the electric field by the following equation:

$$\vec{E} = -\vec{\nabla}\phi$$

With the help of the divergence of the electric displacement density field $\vec{D}$ and in the case of spatial constant permittivity ε we find

$$\vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon\vec{E}) = -\varepsilon\vec{\nabla} \cdot (\vec{\nabla}\phi) = \rho$$

This results in Poisson's equation

$$\vec{\nabla}^2\phi = -\frac{\rho}{\varepsilon} \tag{1.10}$$

If there exits no charge density distribution, this equation reduces to Laplace's equation

$$\vec{\nabla}^2\phi = 0 \tag{1.11}$$

The basic problem of electro static fields is to slove Piosson's or Laplace's equation for a potential function ϕ satisfying appropriate boundary conditions to be discussed later. To get familiare with Piosson's equation we will study the one dimensionale pn-junction as an example.

Example: pn-junction
Figure 1.6a shows a pn-junction where the p and n regions of the semiconductor a seperated by an imaginary sheet. In Figure 1.6b this sheet has been removed. Now the free electrons in the n region will recombinate with the free holes of

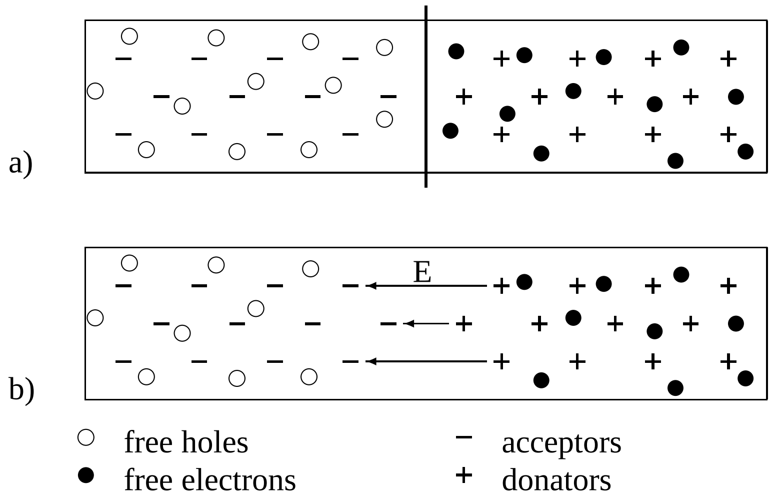

Figure 1.6: pn-junction

the p region, resulting in a space charge distribution as shown in Figure 1.7. N_D and N_A beeing the donator respectivly acceptor concentrations of the doped semiconductor while w_p and w_n are the widths of the depletion zones in the p and n

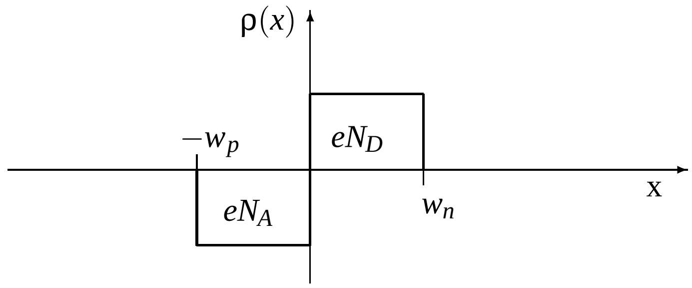

Figure 1.7: charge distribution of a pn-junction

regions. Since the whole semiconductor is neutral, we have the following equation connecting the concentrations of the doping and the depletion width.

$$N_D w_n = N_A w_p$$

To find the electric field distribution and potential function we use the one dimensional divergence equation of the electric displacement density from 1.9.

$$\frac{dE(x)}{dx} = -\frac{e}{\varepsilon} N_A \quad \text{for} \quad -w_p < x < 0$$

$$\frac{dE(x)}{dx} = \frac{e}{\varepsilon} N_D \quad \text{for} \quad 0 < x < w_n$$

Integration yields the following equations for the electric field distribution

$$E(x) = -\frac{eN_A}{\varepsilon}(x+w_p) \quad \text{for} \quad -w_p < x < 0$$

$$E(x) = -\frac{eN_D}{\varepsilon}(w_n - x) \quad \text{for} \quad 0 < x < w_n$$

A second intgeration yields for the potential function

$$\phi(x) = \frac{eN_A}{2\varepsilon}(x+w_p)^2 \quad \text{for} \quad -w_p < x < 0$$

$$\phi(x) = \frac{eN_D}{2\varepsilon}(w_n(w_p+2x) - x^2) \quad \text{for} \quad 0 < x < w_n$$

Figure 1.8 shows the calcualted electric field and potential function of a pn junction.

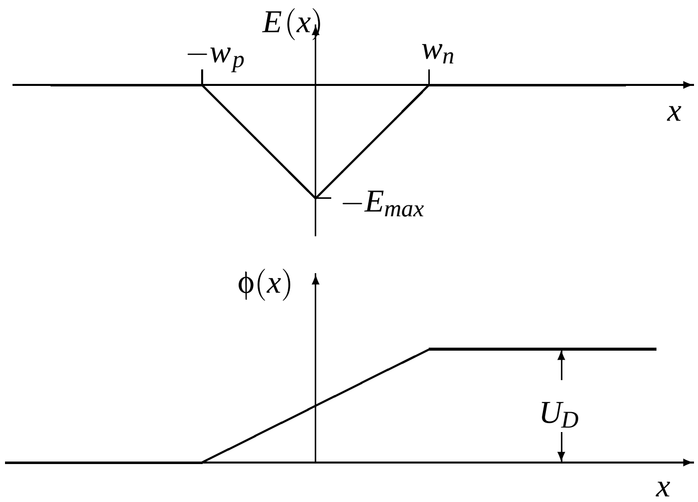

Figure 1.8: Electric field and potential function of a pn-junction

1.3.2 Magnetostatics

Since $\vec{B}$ always has zero divergence, it may be derived from the curl of a vector potential, normally denoted by $\vec{A}$.

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

As a consequence, this makes the divergence of $\vec{B}$ vanish identically. With the curls of the magnetic field $\vec{H}$ and μ beeing spatial constant we find

$$\vec{\nabla} \times (\mu \vec{H}) = \vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu \vec{J}$$

To go one with our considerations we have to use the following identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$$

Since the $\vec{B}$ field defines onyl the curls of $\vec{A}$ its divergence may be set to zero without affecting its curls and hence the equation reduces to

$$\vec{\nabla}^2 \vec{A} = -\mu \vec{J} \tag{1.12}$$

If the problem can be described by cartesian coordinates the last equation can be rewriten in three not coupled equations for all three coordinates.

$$\begin{aligned} \vec{\nabla}^2 A_x &= -\mu J_x \\ \vec{\nabla}^2 A_y &= -\mu J_y \\ \vec{\nabla}^2 A_z &= -\mu J_z \end{aligned} \tag{1.13}$$

Comparing Piossion's eqution of the electrostatic field 1.10 with equation 1.13 clearly shows that electrostatic field problems are easier to solve compared to magnetostatic field problems.

Example

1.4 Wave equation

To gone on in our study of Maxwell's equations we will first consider them in free spaces, this means that there are no charge distributions $\rho \equiv 0$ and no conduction currents $\vec{J} \equiv 0$. On the other hand we will start with arbitrary time dependence. This means we study the possible solutions in the time domain.

1.4.1 Time domain

Under the discussed circumstances the curl equations reduce to the following form

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

We will consider only spatial constant linear lossles isotropic materials, so that the following constitutive relations hold true

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E} \qquad \vec{B} = \mu_0 \mu_r \vec{H} = \mu \vec{H}$$

To find a differential equation we first consider the curls of the first curl equation

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{\partial \vec{\nabla} \times \vec{B}}{\partial t} = -\mu \frac{\partial \vec{\nabla} \times \vec{H}}{\partial t}$$

With the known result of a double curl of a vector field and the help of the second curl equation we find

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E} = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

Since there are no free charges we have $\vec{\nabla} \cdot \vec{E} = 0$ and the first term vanihes. As a result we obtain the so called wave equation.

$$\vec{\nabla}^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \tag{1.14}$$

To find a solution for equation 1.14 we first will consider the special case that only a x-component of the electric field exists which has only a variation in the z-direction.

$$\vec{E} = E_x(z)\vec{e}_x$$

In this special case equation 1.14 reduces to

$$\frac{\partial^2 E_x}{\partial z^2} - \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = 0 \tag{1.15}$$

Following d'Alembert we consider an arbitrary function $E_x(z,t)$ of the following form

$$E_x(z,t) = E_0 \, f(z \pm vt) = E_0 \, f(y)$$

that means, the field distribution is define by an arbitrary function $f(y)$ with the only restriction that its argument has a spezial dependence on the spatial coordinate z and on the time coordinate t with $y = z \pm vt$. For the derivate with respect to z and t we get

$$\frac{\partial^2 E_x}{\partial z^2} = E_0 \frac{\partial^2 f}{\partial y^2} \qquad \frac{\partial^2 E_x}{\partial t^2} = E_0 \, v^2 \frac{\partial^2 f}{\partial y^2}$$

Introducing the above result in equation 1.15 we get

$$(1 - \mu\varepsilon v^2) E_0 \frac{\partial^2 f}{\partial y^2} = 0$$

We have a non trivial solution only in the case were the bracket term vanihes. Hence we find

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{1.16}$$

To discuss the physical meaning of the found solution we have look on Figure 1.9. It shows an arbitrary field distribution $E_x(z,t)$ at the time $t = 0$. If we for example consider the specail point $E_x(0,0) = E_x(y = 0)$ than this point will move in the posistiv or negative z-direction, depending on the sign of v.

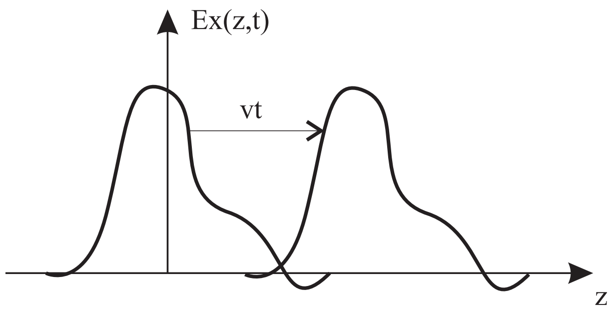

Figure 1.9: Propagation of a wave

$$0 = z \pm vt \quad \rightarrow \quad z = \mp vt$$

So we found that a function of the form $f(z - vt)$ describes a wave with arbitrary form moving in the positive z-direction, whereas a function of the form $f(z + vt)$ descirbes the same have moving in the negative z-direction. If we consider a wave in free space we have $\mu_r = 1$, $\varepsilon_r = 1$ and equation 1.16 reduces to

$$v = c_0 = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \tag{1.17}$$

Equation 1.17 was a create theroretical success of Maxwell's theorie since it shows that electromagnetic waves propagate with the speed of light c_0 and its value could be evaluated from fundamental quantities measurable in the electrostatic and magnetostatic fields. In 1986 the value of the speed of light in free space as fundamental constant was defined to be $c_0 = 2,99792458 \cdot 10^8$m/s by the Task Group on Fundamental Constants, a committee of the International Council of Scientific Unions. For the most cases in the domain of microwaves and optics we have $\mu_r = 1$, so the velocity c of electromagnetic waves in media can be evaluated by

$$c = \frac{c_0}{\sqrt{\varepsilon_r}} = \frac{c_0}{n} \qquad n = \sqrt{\varepsilon_r} \quad \text{refractive index of the media} \tag{1.18}$$

Energy and pointing vector in time domain

To study the behaviour of the electromagnetic field in a general form we will examine the divergence of the cross product $\vec{E} \times \vec{H}$. With the help of the following identity

$$\vec{\nabla} \cdot (\vec{E} \times \vec{H}) \;=\; (\vec{\nabla} \times \vec{E}) \cdot \vec{H} - (\vec{\nabla} \times \vec{H}) \cdot \vec{E}$$

With the help of Maxwell's equations 1.7 the right terms of the last equation may be rewriten as

$$-\frac{\partial \vec{B}}{\partial t} \cdot \vec{H} - (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot \vec{E}$$

Finaly we find for the divergence of the cross product $\vec{E} \times \vec{H}$

$$-\vec{\nabla} \cdot (\vec{E} \times \vec{H}) \;=\; \vec{J} \cdot \vec{E} \;+\; \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{B} \cdot \vec{H}\right) \;+\; \frac{\partial}{\partial t}\left(\frac{1}{2}\vec{D} \cdot \vec{E}\right) \qquad (1.19)$$

To go on in the discussion of equation 1.19 it is nessary to indentify the physical meaning of the right terms. These are

$\vec{J} \cdot \vec{E}$	-	dissipation energy density
$1/2\vec{B} \cdot \vec{H}$	-	energy density of the magnetic field w_m
$1/2\vec{D} \cdot \vec{E}$	-	energy density of the electric field w_e

If we introduce the vector $\vec{S} = \vec{E} \times \vec{H}$ than equation 1.19 definece the divergence of this vector. To come to a physical interpretation we will do an integration over an arbitrary volume that is bounded by a surface A. With the help of the divergence theorem of Gauss the volume integration over $\vec{\nabla} \cdot \vec{S}$ can be transformed in an integration over the bounding surface and we get

$$-\oint_A \vec{S} \cdot d\vec{A} \;=\; \int_V \vec{J} \cdot \vec{E}\, dV \;+\; \frac{\partial}{\partial t}\int_V w_m\, dV \;+\; \frac{\partial}{\partial t}\int_V w_e\, dV$$

Rearranging the last equation we find

$$-\frac{\partial}{\partial t}(W_e + W_m) \;=\; \oint_A \vec{S} \cdot d\vec{A} + \int_V \vec{J} \cdot \vec{E}\, dV \qquad (1.20)$$

with W_m and W_e beeing the total magnetic respectively electric field energie stored in the considered Volume.

$$W_m \;=\; \int_V w_m\, dV \qquad W_e \;=\; \int_V w_e\, dV$$

Considering Figure 1.4.1 equation 1.4.1 states that the time dependent decrease of electric as well as magnetic energie is equal to the total power dissipated in the volume and a power defined by the vector $\vec{S}$ leaving the volume. The vector $\vec{S}$ is denoted as Pointing vector, it descirbes the time dependent power that leaves the volume as radiation.

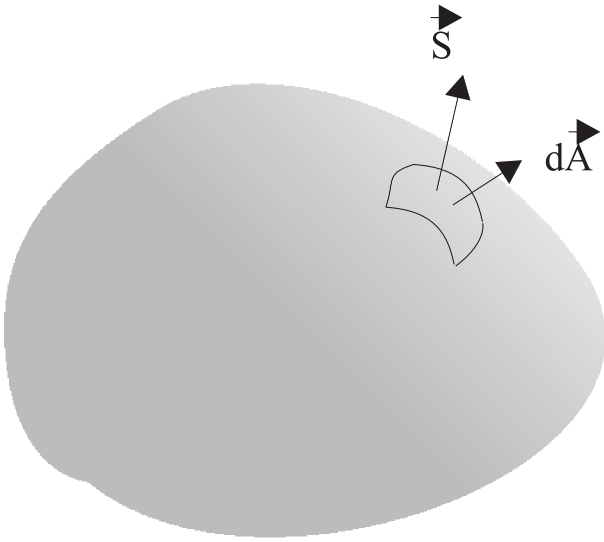

Figure 1.10: Volume of integration to define the pointing vector

1.4.2 Frequency domain

In this section we will consider sinusoidal time dependence. In this case the fields are discribed by phasors defined in 1.1. The differentiation with respect to time in this case is replaced by a multiplication with $j\omega$. So the wave equation 1.14 becomes

$$\vec{\nabla}^2 \underline{\vec{E}} + \omega^2 \mu\varepsilon \underline{\vec{E}} = 0 \tag{1.21}$$

or if we again restrict the electric field to be of the simple form $\underline{\vec{E}} = \underline{\vec{E}}_0(z)\vec{e}_z$ equation 1.21 reduces to

$$\frac{d\underline{E}_x(z)}{dz} + \omega^2 \mu\varepsilon \underline{E}_x = 0$$

That is a second order homogenous linear differential equation with constant coefficient. It is well known that this differential equation is solved by the exponential funktion

$$\underline{E}_x(z) = \underline{E}_0 \exp(-jkz) \quad \text{with} \quad k = \omega\sqrt{\mu\varepsilon} \tag{1.22}$$

The introduced constant k is called the wave number. To get a physical interpretation of equation 1.25 we have to convert the phasor representation to really fields. This is done by multiplying equation **??** with $\exp(j\omega t)$ and considering only the real part.

$$E_x(z,t) = \mathbf{Re}\{\underline{E}_x(z)\exp(j\omega t)\} = E_0 \cos(\omega t - kz)$$

In the last equation we consider the constant $\underline{E}_0$ to be real. Figure 1.11 shows the field distribution of the electric field along the z-coordinate at $t = 0$ and after a

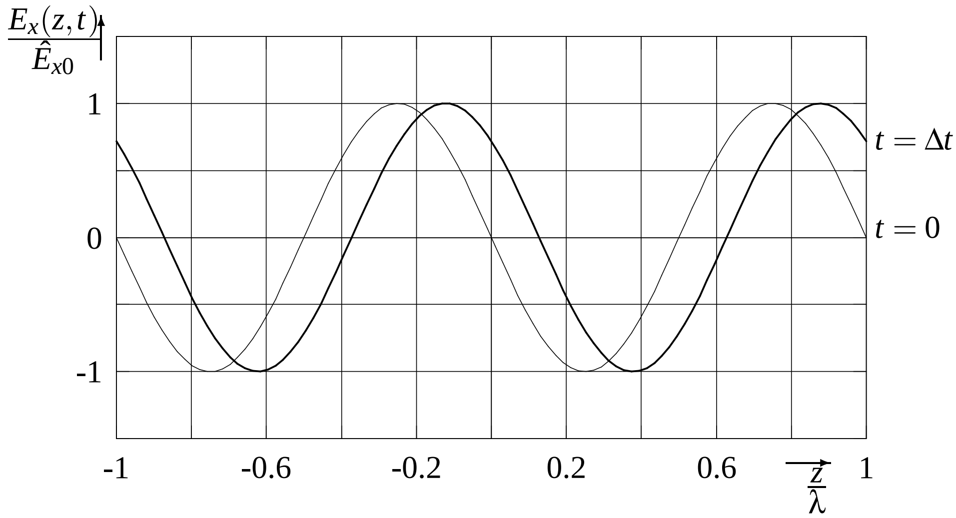

Figure 1.11: Electric field distribution

time intervall Δt. In contrast to Figure 1.9 the electric field extens form $z \to -\infty$ to $z \to \infty$. That accounts for the fact, that in the sinusoidal case the sources are supposed to radiate since $t \to -\infty$. From Figure 1.11 it is also clear that the function $\exp(-jkz)$ describes a wave moving in the posistive z-direction. If we investigate the the total phase of the cosine function $\varphi = \omega t - kz$ and if we for example examine a point with constant phase $\varphi = 0$, we find for the phase velocity

$$0 = \omega t - kz \quad \to \quad z = \frac{\omega}{k} t = ct$$

From the above equation we deduce that a point of constant phase is moving with c the velocity of light. Another important constant of a wave with sinusiodal time dependence is the wavelength λ, that for example the distance between to hills of the wave. With the help of Figure 1.11 we find

$$\lambda = \frac{c}{f} \tag{1.23}$$

Energy and pointing vector in frequeny domain

Hier soll noch was hin, siehe Olver

1.5 Boundary conditions

In order to find proper and unique solutions to Maxwell's equation for situations of pratical interest a knowledge of the behavior of the electric and magnetic field at boundaries separating differnent material bodies is required.

The integral formulation of Maxwell's equation provide the most convenient formulation in order to deduce the required boundary conditions. Consider two media with parameters ε_1, μ_1 and ε_2, μ_2 form a boundary as shown in Figure 1.12. To deduce boundary conditions for the electrical field we first consider a small

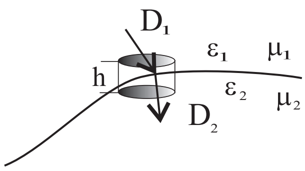

Figure 1.12: Boundary between two different media

cylinder of height $h \to 0$, if we now apply Gauss's law we find in the case of vansihing surface charges.

$$\lim_{h \to 0} \oint_C \vec{D} \cdot d\vec{A} \; = \; 0$$

As the height of the cylinder tends to zero only the top and botton surface of the cylinder will contribut to the integral and we find

$$\vec{D}_1 \cdot \Delta\vec{A}_1 + \vec{D}_2 \cdot \Delta\vec{A}_2 \; = \; 0$$

If we now remember that the top and the botton surface have the same value ΔA, but their normal unit vectors have opposite direction $\vec{e}_1 = -vecu_2 = vecu_n$ we find

$$\vec{D}_1 \cdot \vec{e}_n \Delta A \; - \; \vec{D}_2 \cdot \vec{e}_n \Delta A \quad \to \quad \vec{D}_1 \cdot \vec{e}_n \; = \; \vec{D}_2 \vec{e}_n$$

The last equation states that the normal conponent of the electric displacement density has to be continous at a boundary interface. A similar result clearly holds true for the magnetic flux density.

$$\oint_C \vec{B} \cdot d\vec{A} \; = \; 0 \quad \to \quad \vec{B}_1 \cdot \vec{e}_n \; = \; \vec{B}_2 \cdot \vec{e}_n$$

To obtain boundary conditions on the tangential components of the electric field $\vec{E}$ and the magnetic field $\vec{H}$ the circulation integrals as shown in Figure 1.13 are used. To deduce the condition we define a tangent unit vector $\vec{e}_t$ lying in the boundary

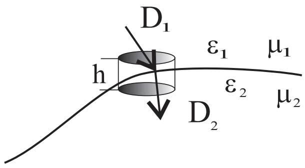

Figure 1.13: Boundary between two different media

plane as well as in the plane of the circulation integral. Again we suppose that the height h of the last plane tends to zero and so does the magnetic flux it. So we find

$$\lim_{h\to 0}\oint_C \vec{E}\cdot d\vec{s} \approx \vec{E}_1\cdot\Delta\vec{s}_1 + \vec{E}_2\cdot\Delta\vec{s}_2 = 0$$

If we take into consideration the different orientations of the wayelements $\Delta\vec{s}_1 = -\Delta\vec{s}_2 = \Delta s\vec{e}_t$ we get

$$\vec{E}_1\cdot\vec{e}_t\Delta s - \vec{E}_2\cdot\vec{e}_t\Delta s = 0 \quad\rightarrow\quad \vec{E}_1\cdot\vec{e}_t = \vec{E}_2\cdot\vec{e}_t$$

The last equations states that the tangential components of the electric field have to be continous at a boundary interface. The same result we find for the tangential magnetic field from its circulation integral.

$$\lim_{h\to 0}\oint \vec{H}\cdot d\vec{s} = 0 \quad\rightarrow\quad \vec{H}\cdot\vec{e}_t = \vec{H}\cdot\vec{e}_t$$

The boundary conditions at a conducting surface will be discussed in the next section.

1.6 Plane wave

We will now go on to discuss one solution of Maxwell's equation the plane wave allready started in 1.4 in a more generall form. In the time harmonic case the wave equation 1.21 reads

$$\vec{\nabla}^2\underline{\vec{E}} + k^2\underline{\vec{E}} = 0 \quad\text{with}\quad k = \omega\sqrt{\mu\varepsilon}$$

Hence the electric field is a solution of the Helmholtz equation. This vector equation holds true for each component and in a cartesian coordinate system we get

$$\begin{aligned}\vec{\nabla}^2\underline{E}_x(x,y,z) &+ k^2\underline{E}_x(x,y,z) = 0\\ \vec{\nabla}^2\underline{E}_y(x,y,z) &+ k^2\underline{E}_y(x,y,z) = 0\\ \vec{\nabla}^2\underline{E}_z(x,y,z) &+ k^2\underline{E}_z(x,y,z) = 0\end{aligned}$$

That are three partial differential equations for the three unknown functions $\underline{E}_x$, $\underline{E}_y$ and $\underline{E}_z$. A standard procedure for solving a partiall differential equation is the method of *separation of variables*. However, this method does not work for all types of partial differential equations in all various coordinate systems. But in our case it will work. The basic method is to express the unknown functions by a product of funktions depending only on one variable. We will discuss this in detial for the function $\underline{E}_x$

$$\underline{E}_x(x,y,z) \;=\; \underline{X}(x)\underline{Y}(y)\underline{Z}(z)$$

Substituting this expression intio the wave equation yields

$$\frac{\partial^2 \underline{X}}{\partial x^2}\underline{Y}\,\underline{Z} + \underline{X}\frac{\partial^2 \underline{Y}}{\partial y^2}\underline{Z} + \underline{X}\,\underline{Y}\frac{\partial^2 \underline{Z}}{\partial z^2} + k^2 \underline{X}\,\underline{Y}\,\underline{Z} \;=\; 0$$

Dividing the last equation by the total function $\underline{E}_x$ gives

$$\frac{1}{\underline{X}(x)}\frac{\partial^2 \underline{X}}{\partial x^2} + \frac{1}{\underline{Y}(y)}\frac{\partial^2 \underline{Y}}{\partial y^2} + \frac{1}{\underline{Z}(z)}\frac{\partial^2 \underline{Z}}{\partial z^2} + k^2 \;=\; 0$$

Each of the first three terms is a function of only one single independent variable and hence the sum of these terms can equal a constant $-k^2$ only if and only if each term is equal to a constant. Thus the partial differential equation of three unknown funktions is separated in three ordinary differential equations of only one unknown function each

$$\begin{aligned} \frac{1}{\underline{X}(x)}\frac{d^2 \underline{X}}{dx^2} &= -k_x^2 \\ \frac{1}{\underline{Y}(y)}\frac{d^2 \underline{Y}}{dy^2} &= -k_y^2 \\ \frac{1}{\underline{Z}(z)}\frac{d^2 \underline{Z}}{dz^2} &= -k_z^2 \end{aligned} \qquad (1.24)$$

with the so called separation condition

$$k_x^2 + k_y^2 + k_z^2 \;=\; k^2$$

The differential equations of 1.24 may all be solved by the exponential function, so we find the following solution for unknown field $\underline{E}_x(x,y,z)$

$$\underline{E}_x(x,y,z) \;=\; \underline{E}_{x0}\exp(-jk_x x)\exp(-jk_y y)\exp(-jk_z z)$$

If we introduce a wave vector $\vec{k}$ by

$$\vec{k} \;=\; k_x\vec{e}_x + k_y\vec{e}_y + k_z\vec{e}_z$$

and if we introduce the vector $\vec{r}$ to the position of an arbitrary point in space

$$\vec{r} \;=\; x\vec{e}_x + y\vec{e}_y + z\vec{e}_z$$

we can rewrite the equation for the field component function $\underline{E}_x$ in the following form

$$\underline{E}_x(\vec{r}) \;=\; \underline{E}_{x0}\exp(-j\vec{k}\cdot\vec{r})$$

Similar solutions may be found for the field components $\underline{E}_y$ and $\underline{E}_z$

$$\begin{aligned}\underline{E}_y(\vec{r}) &= \underline{E}_{y0}\exp(-j\vec{k}\cdot\vec{r})\\ \underline{E}_z(\vec{r}) &= \underline{E}_{z0}\exp(-j\vec{k}\cdot\vec{r})\end{aligned}$$

In a full vector formulation the found solution for the Helmholz equation for a space described by the parameters μ and ε reads

$$\underline{\vec{E}}(\vec{r}) \;=\; \underline{\vec{E}}_0\exp(-j\vec{k}\cdot\vec{r}) \tag{1.25}$$

In generall the constant vector $\underline{\vec{E}}_0$ may be a complex vector, to simplify the following considerations we will suppose that it is real. Since we did not consider any charge density in that space the above field has to fullfill the divergence condition $\vec{\nabla}\cdot\underline{\vec{E}}(\vec{r}) = 0$ or with the help of equation 1.25 we find

$$\vec{\nabla}\cdot[\vec{E}_0\exp(-j\vec{k}\cdot\vec{r})] \;=\; \vec{E}_0\cdot[\vec{\nabla}\cdot\exp(-j\vec{k}\cdot\vec{r})] \;=\; -j\vec{k}\cdot\vec{E}_0\exp(-j\vec{k}\cdot\vec{r}) \;=\; 0$$

Hence we have

$$\vec{k}\cdot\vec{E}_0 \;=\; 0 \tag{1.26}$$

that means the constant vector $\vec{E}_0$ has to be perpendicular to the wave vector $\vec{k}$. A solution for the magnetic field can be found by using the curl equation of the electric filed from 1.8, which leads to

$$\underline{\vec{H}} \;=\; -\frac{1}{j\omega\mu}\vec{\nabla}\times[\vec{E}_0\exp(-j\vec{k}\cdot\vec{r})] \;=\; \frac{1}{j\omega\mu}\vec{E}_0\times[\vec{\nabla}\cdot\exp(-j\vec{k}\cdot\vec{r})] \;=$$

$$\frac{1}{\omega\mu}\vec{k}\times\vec{E}_0\exp(-j\vec{k}\cdot\vec{r})$$

If we introduce a unit vector $\vec{e}_w$ in the direction the wave is propagting we have $\vec{k} = k\vec{e}_w$ and find as final equation for the magnetic field

$$\underline{\vec{H}} \;=\; \frac{k}{\omega\mu}\vec{e}_w\times\vec{E}_0\exp(-j\vec{k}\cdot\vec{r}) \tag{1.27}$$

It is interesting to examine the inverse of first term in more detail. With equation 1.21 we defined k to be equal to $\omega\sqrt{\mu\varepsilon}$. Hence we get

$$\frac{\omega\mu}{k} = \sqrt{\frac{\mu_r\,\mu_0}{\varepsilon_r\,\varepsilon_0}}$$

One ususally defines the so called intrinsic impedance Z_0 of free space with

$$Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\Omega \qquad (1.28)$$

With the help of this constant one defines the field impedane of a plane wave in a space defined by the μ_r and ε_r by:

$$Z_F = \sqrt{\frac{\mu_r}{\varepsilon_r}} Z_0$$

The inverse of this value is called field admittance $Y_F = 1/Z_F$. Thus the equation for the megnatic filed reads

$$\underline{\vec{H}} = \frac{1}{Z_F}\vec{e}_w \times \underline{\vec{E}}(\vec{r}) \qquad (1.29)$$

Note that $\underline{\vec{H}}$ is perpendicular to the electric field $\underline{\vec{E}}$ and the dircetion of wave propagation $\vec{e}_w$. Hence both the electric field and the magnetic field lie in constant phase planes. For this reason this type of wave is called a **t**ransverese **e**lectro**m**agnetic wave (TEM-wave).

Real electric field

To obtain the real electric field corresponding to the phasor represtentation of equation 1.25 we have to do the following operation

$$\vec{E}(\vec{r},t) = \mathbf{Re}[\vec{E}_0 \exp(-j\vec{k}\cdot\vec{r})\exp(j\omega t)] = E_0\cos(\vec{k}\vec{r} - \omega t)$$

The wavelength again is the distance the wave must travel to undergo a phase change of 2π. Thus we find

$$k\lambda = 2\pi \quad \rightarrow \quad \lambda = \frac{c}{f}$$

with c beeing the speed of light in a media.

$$c = \frac{c_0}{\sqrt{\mu_r\varepsilon_r}}$$

Phase velocity

The phase velocity is the velocity with which an observer would have to move in order to see a constant phase of the wave. For the phase to be constant we have

$$\vec{k}\cdot\vec{r} - \omega t \;=\; const.$$

If we introduce θ as an angle between the direction of wave propagation $\vec{k}$ and $\vec{r}(t)$ the direction of movement we have

$$k\,r(t)\,\cos(\theta) \;-\; \omega t \;=\; const.$$

Differentiation of the last equation with respect to time yields for the phase velocity v_{Ph}

$$v_{Ph} \;=\; \frac{dr}{dt} \;=\; \frac{\omega}{k\cos(\theta)} \;=\; \frac{c}{\cos(\theta)} \tag{1.30}$$

It is intereseting to notice that only in the direction of wave propagation ($\theta = 0$) we have a phase velocity which is equal to the speed of light but in all other directions the phase velocity is greater.

Power density

To calculate the power density that is transported by a plane wave we evaluate the time averaged pointing vector

$$<\vec{S}> = \frac{1}{2}\mathbf{Re}[\underline{\vec{E}}\times\underline{\vec{H}}^{*}] \tag{1.31}$$

With the help of equation 1.29 we find

$$<\vec{S}> = \frac{1}{2Z_F}\mathbf{Re}[\underline{\vec{E}}\times(\vec{e}_w\times\underline{\vec{E}}^{*})] \;=\; \frac{1}{2Z_F}\mathbf{Re}[\vec{e}_w(\underline{\vec{E}}\cdot\underline{\vec{E}}^{*}) - \underline{\vec{E}}^{*}(\underline{\vec{E}}\cdot\vec{e}_w)]$$

Since the last term vanishes we get for the power density transported by a plane wave

$$<\vec{S}> = \frac{1}{2Z_F}|\underline{\vec{E}}|^2\vec{e}_w \tag{1.32}$$

Of course the power density is transported in the direction of wave propagation.

1.6.1 Reflection from a dielectric interface

In Figure 1.14 the half space $z \geq 0$ is filled with a dielectric media with total permitivity ε_2. A plane wave is assumed to incident from the region $z < 0$. Without loss of generality the xy-plane is orientated so that the unit vector $\vec{e}_i$ spezifying

Figure 1.14: Reflection from a dieletric boundary

the direction of the incident wave lies in the xy-plane. Then its direction can be defined with the help of the unit vectors $\vec{e}_x$, $\vec{e}_z$ and the angle θ_i defined in Figure 1.14.

$$\vec{e}_i \;=\; \vec{e}_x \sin(\theta_i) + \vec{e}_z \cos(\theta_i)$$

Parallel polarization

We first will consider the case were the electric field vector is lying coplanar with $\vec{e}_i$ in the xz-plane. With the help of the propagation vector of the plane wave, ist wave vector $\vec{k}_i$ is given by.

$$\vec{k}_i \;=\; \frac{2\pi}{\lambda_1}\vec{e}_i \;=\; \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_r}\,\vec{e}_i$$

So the electric and magnetic fields of the incident wave are described by the following equation

$$\underline{\vec{E}}_i(\vec{r}) \;=\; \vec{E}_{i0}\exp(-j\vec{k}_i\cdot\vec{r}) \qquad \underline{\vec{H}}_i(\vec{r}) \;=\; Y_{F1}\,\vec{e}_i \times \underline{\vec{E}}_i(\vec{r})$$

Part of the incident power will be reflected and the remeinder will be transmitted in to the dielectric media. To describe the direction of the reflected wave we use the unit vector $\vec{e}_r$ which is defined with the help of θ_r shown in Figure 1.14.

$$\vec{e}_r \;=\; \vec{e}_x \sin(\theta_r) - \vec{e}_z \cos(\theta_r)$$

The electric field as well as the magnetic field are to define analog to the incident wave,

$$\underline{\vec{E}}_r(\vec{r}) \;=\; \vec{E}_{r0}\exp(-j\vec{k}_r\cdot\vec{r}) \qquad \underline{\vec{H}}_r(\vec{r}) \;=\; Y_{F1}\,\vec{e}_r \times \underline{\vec{E}}_r(\vec{r})$$

with

$$\vec{k}_r \;=\; \frac{2\pi}{\lambda_1}\vec{e}_r \;=\; \frac{2\pi}{\lambda_0}\sqrt{\varepsilon_r}\,\vec{e}_r$$

In the dielectric media 2 the solution for the plane wave is the same as in medium 1 but with ε_1 replaced by ε_2. To define the direction of propagation in Figure 1.14 a third angle θ_t was defined. So we get for the propagation vector and the fields of the transmitted wave.

$$\vec{e}_t \;=\; \vec{e}_x \sin(\theta_t) + \vec{e}_z \cos(\theta_t)$$

$$\underline{\vec{E}}_t(\vec{r}) \;=\; \vec{E}_{t0} \exp(-j\vec{k}_t \cdot \vec{r}) \qquad \underline{\vec{H}}_t(\vec{r}) \;=\; Y_{F2}\, \vec{e}_t \times \underline{\vec{E}}_t(\vec{r})$$

At the moment the two amplitudes E_{r0}, E_{t0} and the angle of refelction θ_r and transmission θ_t are unknown. To solve for this values he have to apply the boundary condistions already discussed. That means the tangential components of the electric and magnetic fields have to be continous at the interface $z = 0$. Of course these components have to be continous for all values of x and y in this plane. This is only possible if the fields on adjacent sides of the boundary have the same variation with x and y. Hence we must have

$$k_i\, e_{ix} \;=\; k_r\, e_{rx} \;=\; k_t\, e_{tx}$$

The first part leads to the following conglution

$$\frac{2\pi}{\lambda_1} \sin(\theta_i) \;=\; \frac{2\pi}{\lambda_1} \sin(\theta_r) \quad \to \quad \theta_i \;=\; \theta_r$$

This is the well known law of reflection allready known from playing pool billiard. In the second case we have to fullfill the following equation

$$\frac{2\pi}{\lambda_1} \sin(\theta_i) \;=\; \frac{2\pi}{\lambda_2} \sin(\theta_t)$$

With the help of the refraction coefficients $n_1 = \sqrt{\varepsilon_1}$ and $n_2 = \sqrt{\varepsilon_2}$ the last equation may be rewritten as

$$\frac{2\pi}{\lambda_0} n_1 \sin(\theta_i) \;=\; \frac{2\pi}{\lambda_0} n_2 \sin(\theta_t)$$

This leads directly to Snell's law of refraction

$$\frac{\sin(\theta_t)}{\sin(\theta_i)} \;=\; \frac{n_1}{n_2} \tag{1.33}$$

Example for total reflection

To go on in our discusion we have to consider the field components in the interface. In the following considerations we will use $\theta_1 = \theta_i = \theta_r$ and $\theta_2 = \theta_t$.

For the electric field x-components of the incident, the reflected and transmitted wave we find

$$\begin{aligned} E_{ix} &= E_{i0}\cos(\theta_1) \\ E_{rx} &= E_{r0}\cos(\theta_1) \\ E_{tx} &= E_{t0}\cos(\theta_2) \end{aligned}$$

Imposing the boundary condition of continuity to the x component at $z=0$ yileds the following relation

$$(E_{i0}+E_{r0})\cos(\theta_1) \;=\; E_{t0}\cos(\theta_2)$$

Which can be brought to the following form with the help of Snell's law 1.33

$$(E_{i0}+E_{r0})\cos(\theta_1) \;=\; E_{t0}\sqrt{1-\sin^2(\theta_2)} \;=\; E_{t0}\sqrt{1-\frac{n_1}{n_2}\sin^2(\theta_1)}$$

Of course the magnetic field has only y-components and hence we find for

$$H_{iy} \;=\; Y_{F1}E_{i0} \quad H_{ry} \;=\; -Y_{F2}E_{r0} \quad H_{ty} \;=\; Y_{F2}E_{t0}$$

Thus the continuity of the tangential magnetic field imposes the following equation.

$$Y_{F1}(E_{i0}-E_{r0}) \;=\; Y_{F2}E_{t0}$$

If we define a reflection coefficient by $r=E_{r0}/E_{i0}$ and a transmission coeffcient by $t=E_{t0}/E_{i0}$ the two boundary conditions result in the following two equations

$$\begin{aligned} (1+r)\cos(\theta_1) &= t\sqrt{1-\tfrac{n_1}{n_2}\sin^2(\theta_1)} \\ n_1(1-r) &= n_2 t \end{aligned}$$

If we solve this equations for the reflection and transmission coefficient we find

$$r \;=\; \frac{\sqrt{\left(\frac{n_2}{n_1}\right)^2-\sin^2(\theta_1)}-\left(\frac{n_2}{n_1}\right)^2\cos(\theta_1)}{\sqrt{\left(\frac{n_2}{n_1}\right)^2-\sin^2(\theta_1)}+\left(\frac{n_2}{n_1}\right)^2\cos(\theta_1)} \tag{1.34}$$

$$t \;=\; \frac{2\left(\frac{n_2}{n_1}\right)^2\cos(\theta_1)}{\sqrt{\left(\frac{n_2}{n_1}\right)^2-\sin^2(\theta_1)}+\left(\frac{n_2}{n_1}\right)^2\cos(\theta_1)} \tag{1.35}$$

Figure 1.15: Reflection from a dieletric boundary

Perpendicular polarization

In contrast to the first case of parallel polariziazed electric field we will discuss the case of perpendicular polarization not in detail. Figure 1.15 shows the consider configuration. As shown in this figure now the electric field vector is perpendicular to plane of incident. This leads to different equations to fullfill the boundary conditions. As a consequence the equations for the reflection and transmission coefficient differ from the parallel case.

$$r = \frac{\cos(\theta_1) - \sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2(\theta_1)}}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2(\theta_1)} + \cos(\theta_1)} \tag{1.36}$$

$$t = \frac{2\cos(\theta_1)}{\sqrt{\left(\frac{n_2}{n_1}\right)^2 - \sin^2(\theta_1)} + \cos(\theta_1} \tag{1.37}$$

Example for reflections Brewster angle

1.6.2 Reflection from a conducting plane

To study the reflection from a conducting plane we consider Figure 1.17. The half space $z \geq 0$ is filled with a conducting material with conductance σ. A plane wave with the electric field vector polarized in the x-direction is incident perpendicular on that plane coming from $z \rightarrow -\infty$. Of course a reflected wave will emerge from the plane and propagate in the negative z-direction. If we again define the

Figure 1.16: Reflection coefficient as function of angle of incident

Figure 1.17: Reflection from a conducting plane

reflection coefficient as the ratio of reflected to incident field amplitude of the electric field, we have the following fields in the half space $z < 0$.

$$\begin{aligned}
\underline{\vec{E}}_i(z) &= E_{i0}\vec{e}_x \exp(-jk_0 z) \\
\underline{\vec{H}}_i(z) &= Y_0 E_{i0}\vec{e}_y \exp(-jk_0 z) \\
\underline{\vec{E}}_i(z) &= \underline{r} E_{i0}\vec{e}_x \exp(jk_0 z) \\
\underline{\vec{H}}_i(z) &= -\underline{r} Y_0 E_{i0}\vec{e}_y \exp(-jk_0 z)
\end{aligned}$$

In the time harmonic case we found for the complex permitivity

$$\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$

in the case of high conductivity this equation reduces to

$$\underline{\varepsilon} = -j\frac{\sigma}{\omega}$$

and hence the wave equation in the conducting material becomes

$$\vec{\nabla}^2 \underline{\vec{E}} - -j\omega\mu\sigma\underline{\vec{E}} = 0 \tag{1.38}$$

Equation 1.38 has the form of a diffusion equation known from the flow of heat in a thermal conductor. In our case we will only consider a electric field in x-direction with a z-dependence, hence equation 1.38 reduces to an ordinary differential equation

$$\frac{d^2\underline{E}_t(z)}{dz^2} - j\omega\mu\sigma\underline{E}_t(z) = 0$$

It is well known that this differentail equation is solved by the exponential function, so we find as solution

$$\vec{\underline{E}}(z) = \underline{E}_{t0}\vec{e}_x \exp(-\underline{\gamma} z) \quad \text{with} \quad \underline{\gamma} = \sqrt{j\omega\mu\sigma}$$

If we take the square root of j we can break $\underline{\gamma}$ in its real (α) and imaginary (β) part

$$\underline{\gamma} = (1 + j)\sqrt{\frac{\omega\mu\sigma}{2}}$$

If we are only interested in the absolute value of the transmitted electric field we find the following functional dependence

$$|\vec{\underline{E}}_t(z)| = |\underline{E}_{t0}| \exp(-\alpha z)$$

From the last equation we see that the transmitted field decays exponentially in the conducting material. One defines a so called skin depth δ_s as that distance z from the surface, were the absolute value of the electric field is already reduced by a factor $1/e$. This leads to the following equation for δ_s

$$\delta_s = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{1.39}$$

Example: Skin depth for cooper at f = 100MHz
As conductivity for cooper we have $\sigma_{Cu} = 6 \cdot 10^7 \frac{S}{m}$ and as permeabillity we have to us μ_0 Thus we get for the skin depth

$$\delta_s = \sqrt{\frac{2}{2\pi 10^8\, 4\pi 10^{-7}\, 6\, 10^6}} m \approx 6,5\mu m$$

To callculate the magnetic field we have to use the curl equation for the electric field of 1.8

$$\vec{\underline{H}}_t = -\frac{1}{j\omega\mu}\vec{\nabla} \times \vec{\underline{E}}_t = \frac{\underline{\gamma}}{j\omega\mu\sigma}\underline{E}_{t0} \exp(-\underline{\gamma} z)\vec{e}_y \tag{1.40}$$

With the help of equation 1.40 we are able to define the field impedance in a conductor $\underline{Z}_m$

$$\underline{Z}_m = \frac{j\omega\mu}{\underline{\gamma}} = (1 + j)\sqrt{\frac{\omega\mu}{2\sigma}} = (1 + j)\frac{1}{\sigma\delta_s}$$

Example: Field impedance of cooper at 100MHz

$$Z_m = \frac{1+j}{6\,10^7\,6,5\,10^{-6}}\Omega \approx (1+j)2,6m\Omega$$

This is a very small value compared to the field impedance of plane wave in free space.

Returning to the boundary value problem and imposing the boundary conditions of continuity of the tangential fields at the boundary $z = 0$ gives the following equations

$$(1+\underline{r}) = \underline{t}$$
$$(1-\underline{r})Y_0 = \underline{Y}_m\underline{t}$$

Solving this equations for the reflection and transmission coefficient yields

$$\begin{aligned} \underline{r} &= \frac{\underline{Z}_m - Z_0}{\underline{Z}_m + Z_0} \\ \underline{t} &= \frac{2\underline{Z}_m}{\underline{Z}_m + Z_0} \end{aligned} \tag{1.41}$$

Since the absolute value of $\underline{Z}_m$ is very small compared to Z_0 the intrinsic impedance of free space, the reflection coefficient $\underline{r}$ is almost equal to -1 and the transmission coefficient is very small. Almost all the incident power is therefor reflected from metallic boundary. For the electric field in the conductor in dependence of the magnetic field $\underline{H}_{y0}$ on its surface we find

$$\underline{E}_x(z) = (1+j)\sqrt{\frac{\mu\omega}{2\sigma}}\underline{H}_{y0}\exp(-\underline{\gamma}z)$$

In the limit $\sigma \to \infty$ for a perfect conductor the electric field will vanish. This results in the following boundary conditions on a surface of a perfect conductor with surface normal $\vec{e}_n$

$$\begin{aligned} \vec{e}_n \times \underline{\vec{E}} &= 0 \\ \vec{e}_n \times \underline{\vec{H}} &= \underline{\vec{J}}_s \qquad \text{surface current} \end{aligned} \tag{1.42}$$

Muss noch etwas ausführlicher

Chapter 2

Transmission lines and waveguides

In this chapter we will deal with waves that are propagating along three dimensional structures. We will allways suppose that the cross section of the structure will not change in the z-direction, so that waves guided by the structure will propagate for example in the positive z-direction. In this chapter we will not deal

Figure 2.1: Cross section of different wave guides

with reflected waves, because the electromagnetic field distribution of that waves does'nt differ essentially in the considered cross section. Figure 2.1 shows some examples for wave guides we will investigate in this chapter.

2.1 Classification of wave solutions

Since no sources are considered the electric and magnetic fields are solutions of the homogeneous Helmholtz equations

$$\vec{\nabla}^2\underline{\vec{E}} + k^2\underline{\vec{E}} = 0 \quad \text{or} \quad \vec{\nabla}^2\underline{\vec{H}} + k^2\underline{\vec{H}} = 0 \tag{2.1}$$

In this chapter we will try to find solutions of the above equations that describe waves propagating in the positive z-direction of the structure. As we saw in the preceding chapter in the time harmonic case wave propagation in the positive z-direction is describe by the function $\exp(-j\beta z)$ with β beeing the phase propagation factor of the considered wave. To deal with arbitray structures in the xy-plane we first rewrite the nabla operator in a first part working in the transversal plane and a second one working in the z-direction

$$\vec{\nabla} = \vec{\nabla}_t + \vec{e}_z \frac{\partial}{\partial z}$$

All fields considered in this chapter will have a z-dependence describe by $\exp(-j\beta z)$. So the differentiation with respect to z in above nabla operator may be replaced by a multiplication with $-j\beta$. Hence we find its new form

$$\vec{\nabla} = \vec{\nabla}_t + -j\beta\vec{e}_z$$

We will also decompose the electric and the magnetic field into a transversal and an axial component.

$$\underline{\vec{E}}(x,y,t) = \underline{\vec{E}}_t(x,y)\exp(-j\beta z) + \underline{\vec{E}}_z(x,y)\exp(-j\beta z)$$

$$\underline{\vec{H}}(x,y,t) = \underline{\vec{H}}_t(x,y)\exp(-j\beta z) + \underline{\vec{H}}_z(x,y)\exp(-j\beta z)$$

In the above equations the field distribution in the transversal plane is formulatted for simplicity as dependence on the variables x and y, but other transversal coordinate systems are also applyable. We will now rewrite the curl equations of 1.8, to see how they change under the considered circumstances

$$\vec{\nabla} \times \underline{\vec{E}} = [\vec{\nabla}_t - j\beta\vec{e}_z] \times [\underline{\vec{E}}_t(x,y) + \underline{\vec{E}}_z(x,y)]\exp(-j\beta z) =$$

$$-j\omega\mu[\underline{\vec{H}}_t(x,y) + \underline{\vec{H}}_z(x,y)]\exp(-j\beta z)$$

If we consider the different direction, we notice that the above equation may be separated in two ones.

$$\vec{\nabla}_t \times \underline{\vec{E}}_t(x,y) = -j\omega\mu\underline{\vec{H}}_z(x,y) \tag{2.2}$$

$$\vec{e}_z \times \vec{\nabla}\underline{E}_z(x,y) + j\beta\vec{e}_z \times \underline{\vec{E}}_t(x,y) = j\omega\mu\underline{\vec{H}}_t(x,y) \tag{2.3}$$

In analog manner the curl equation of the amgnetic field of 1.8 may be decomposed into the following equations

$$\vec{\nabla}_t \times \underline{\vec{H}}_t(x,y) = j\omega\varepsilon\underline{\vec{E}}_z(x,y) \tag{2.4}$$

$$\vec{e}_z \times \vec{\nabla}\underline{H}_z(x,y) + j\beta\vec{e}_z \times \vec{\underline{H}}_t(x,y) \;=\; -j\omega\varepsilon\vec{\underline{E}}_t(x,y) \tag{2.5}$$

If we examine examine the divergence equation of the magnetic field, we find

$$\vec{\nabla}_t \cdot \vec{\underline{H}}_t(x,y) \;=\; j\beta\underline{H}_z(x,y) \tag{2.6}$$

and in the case of the electric field

$$\vec{\nabla}_t \cdot \vec{\underline{E}}_t(x,y) \;=\; j\beta\underline{E}_z(x,y) \tag{2.7}$$

2.1.1 Transverse electromagnetic waves

In this subsection we will discuss the generall properties of TEM waves, waves that have no field components in the direction of propagation. Hence

$$\underline{E}_z \;\equiv\; 0 \qquad \underline{H}_z \;\equiv\; 0$$

In this case the equations 2.2 to 2.7 reduce to

$$\begin{array}{rcrcl}
\vec{\nabla}_t & \times & \vec{E}_t(x,y) & = & 0 \\
\beta\vec{e}_z & \times & \vec{E}_t(x,y) & = & \omega\mu\vec{H}_t(x,y) \\
\vec{\nabla}_t & \times & \vec{H}_t(x,y) & = & 0 \\
\beta\vec{e}_z & \times & \vec{H}_t(x,y) & = & -\omega\varepsilon\vec{E}_t(x,y) \\
\vec{\nabla}_t & \cdot & \vec{\underline{E}}_t(x,y) & = & 0 \\
\vec{\nabla}_t & \cdot & \vec{\underline{H}}_t(x,y) & = & 0
\end{array} \tag{2.8}$$

As a consequence of equation 2.8a the curls of the transversal electric field $\vec{\underline{E}}_t$ vanishes, this means that it may deduced from a scalar potential function $\phi(x,y)$ defined in the transversal plane.

$$\vec{E}_t(x,y) \;=\; -\vec{\nabla}_t\phi(x,y) \tag{2.9}$$

With the help of equation 2.8b we find for the transversal magnetic field

$$\vec{H}_t \;=\; \frac{\beta}{\omega\mu}\vec{e}_z \times \vec{E}_t(x,y) \tag{2.10}$$

Using the divergence equation 2.8e we find for the potential function

$$\vec{\nabla} \cdot [\vec{\nabla}\phi(x,y)] \;=\; \vec{\nabla}^2\phi(x,y) \;=\; 0$$

From the last equation we see that the scalar potential $\phi(x,y)$ has to fullfill Laplace's equation and the certain boundary conditions to be a appropriate function.

The electric field of the TEM-wave propagating along the structure is then given by

$$\underline{\vec{E}}_t(x,y,z) \;=\; -\vec{\nabla}_t \phi(x,y)\exp(-j\beta z) \tag{2.11}$$

Of course also this field has to statisfy Helmholtz equation 2.1, leading to

$$(\vec{\nabla}_t^2 - \beta^2)\underline{\vec{E}}_t + k^2 \underline{\vec{E}}_t \;=\; 0$$

Using the potential function we can rewrite the above equation

$$\vec{\nabla}_t \left[\vec{\nabla}_t^2 \phi(x,y) + (k^2 - \beta^2)\phi(x,y)\right] \;=\; 0$$

Since the first term is zero the bracket $(k^2 - \beta^2)$ must vanish, giving us the propagation phase constant β of TEM waves

$$\beta \;=\; k \;=\; \omega\sqrt{\mu\varepsilon} \;=\; \frac{2\pi}{\lambda} \tag{2.12}$$

With the help of eqution 2.8b we find for the magnetic field of a TEM wave

$$\underline{\vec{H}}_t(x,y,z) \;=\; \frac{\beta}{\omega\mu}\vec{e}_z \times \underline{\vec{E}}_t(x,y,z) \;=\; \frac{1}{Z_F}\vec{e}_z \times \underline{\vec{E}}_t(x,y,z) \tag{2.13}$$

Form this equation it can easily been recognized that the magnetic field is allways perpenticular to the electric field and as in a plane wave the amplitudes of the electric and magnetic fields are connected by the real field impedance Z_F.

Example: Lossless coaxial transmission line
Figure 2.2 shows the cross section of a coaxial transmission line. It is reasonable to use cylindrical coordinates to describe ist geometry and to formulate the boundary problem. In this coordinate system the square of the tranverse nabla operator is given by

$$\vec{\nabla}^2 \;=\; \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}$$

We will search for a potentail function ϕ that is independent of the φ-coordinate, thus the above equation reducxes to

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi(\rho)}{\partial\rho}\right) \;=\; 0$$

This equation can be easily integrated and gives as its result a first constant C_1

$$\rho\frac{\partial\phi(\rho)}{\partial\rho} \;=\; C_1$$

Figure 2.2: Cross section of a coaxial transmission line

A further integration yields the following equation for the potential function

$$\phi(\rho) \;=\; C_1 \ln(\rho) + C_2$$

The two constants of the last equation may be determined by imposing the boundary conditions on the potential function. In generall the outer conductor of a coaxial line is supposed to have the potential $\phi_o = 0$, while a field between the conductors may only exist if the inner conductor has a differnent potential, say $\phi_i = U_0$. So the potential function has to fullfill the following boundary conditions

$$\begin{aligned}\phi(\rho = d/2) &= U_0 &= C_1 \ln(d/2) + C_2 \\ \phi(\rho = D/2) &= 0 &= C_1 \ln(D/2) + C_2\end{aligned}$$

Solving this two equations for to determine the two constant yields the equation for the potential function

$$\phi(\rho) \;=\; \frac{U_0}{\ln(D/d)} \ln\left(\frac{D}{2\rho}\right)$$

Using equation 2.11 we find for the transversal electric field of the TEM wave

$$\vec{E}_t \;=\; -\vec{e}_\rho \frac{d}{\rho}\left[\frac{U_0}{\ln(D/d)} \ln\left(\frac{D}{2\rho}\right)\right] \;=\; \frac{U_0}{\ln(D/d)}\, \frac{1}{\rho}\, \vec{e}_\rho$$

And for the transverse magnetic field, we find

$$\vec{H}_t \;=\; \frac{1}{Z_F} \vec{e}_z \times \left(\frac{U_0}{\ln(D/d)}\, \frac{1}{\rho}\, \vec{e}_\rho\right) \;=\; \frac{U_0}{Z_F \ln(D/d)}\, \frac{1}{\rho}\, \vec{e}_\varphi$$

So we found the equations describing the transversal electric and magnetic field of the TEM wave propagating on a coaxial transmission line.

2.1.2 TE waves

Transverse electric waves are waves that have no axial electric field component $E_z \equiv 0$ but an axial magnetic field component $H_z \neq 0$. As we will see for TE waves H_z plays the role of a potential function from which allother field components may be deduced. In this case the equations 2.2 to 2.7 reduce to

$$\begin{array}{rcrcl} \vec{\nabla}_t & \times & \vec{E}_t(x,y) & = & -j\omega\mu\underline{\vec{H}}_z \\ \beta\vec{e}_z & \times & \vec{E}_t(x,y) & = & \omega\mu\vec{H}_t(x,y) \\ \vec{\nabla}_t & \times & \vec{H}_t(x,y) & = & 0 \\ \beta\vec{e}_z & \times & \vec{H}_t(x,y) & = & -\omega\varepsilon\vec{E}_t(x,y) + j\vec{e}_z \times \vec{\nabla} \cdot \underline{H}_z \\ \vec{\nabla}_t & \cdot & \underline{\vec{E}}_t(x,y) & = & 0 \\ \vec{\nabla}_t & \cdot & \underline{\vec{H}}_t(x,y) & = & j\beta\underline{\vec{H}}_z \end{array} \tag{2.14}$$

Using Helmholtz's equation 2.1 and decomposing the operators and fields in transverse and axial components we have

$$(\vec{\nabla}_t^2 - \beta^2)(\underline{\vec{H}}_t + \underline{\vec{H}}_z) + k^2(\underline{\vec{H}}_t + \underline{\vec{H}}_z) \;=\; 0$$

The last equation seperates in two independent ones

$$\begin{array}{rcrcl} \vec{\nabla}_t\underline{\vec{H}}_t & + & (k^2 - \beta^2)\underline{\vec{H}}_t & = & 0 \\ \vec{\nabla}_t\underline{H}_z & + & (k^2 - \beta^2)\underline{H}_z & = & 0 \end{array}$$

If we introduce a new constant $k_c^2 = (k^2 - \beta^2)$ we have to solve the following partial differential equation to find a solution for the function $\underline{H}_z(x,y)$

$$\vec{\nabla}^2\underline{H}_z(x,y) + k_c^2\underline{H}_z(x,y) \;=\; 0 \tag{2.15}$$

To find an expresion for $\underline{\vec{H}}_T(x,y)$ we calcuate the curls of equation 2.14c and use a known identity of vector analysis

$$\vec{\nabla}_t \times (\vec{\nabla}_t \times \underline{\vec{H}}_t) \;=\; \vec{\nabla}_t(\vec{\nabla} \cdot \underline{\vec{H}}_t) - \vec{\nabla}^2\underline{\vec{H}}_t \;=\; 0$$

Using equation 2.1.2a and equation 2.14f we are able to rewrite the last one in the following form

$$\vec{\nabla}(-j\beta\underline{H}_z) + kc^2\underline{\vec{H}}_t \;=\; 0$$

and solve it to deduce $\underline{\vec{H}}_t$ from the function $\underline{H}_z$

$$\underline{\vec{H}}_t \;=\; -\frac{j\beta}{k_c^2}\vec{\nabla}_t\underline{H}_z(x,y) \tag{2.16}$$

To find $\underline{\vec{E}}_t$ in terms of $\underline{\vec{H}}_t$ we consider the following vector product

$$\beta\vec{e}_z \times (\vec{e}_z \times \underline{\vec{E}}_t) \;=\; \beta\left[(\vec{e}_z \cdot \underline{\vec{E}}_t)\vec{e}_z - \beta(\vec{e}_z \cdot \vec{e}_z)\underline{\vec{E}}_t\right] \;=\; -\beta\underline{\vec{E}}_t$$

With the help of equation 2.14b we find for the transverse electric field $\underline{\vec{E}}_t$

$$\underline{\vec{E}}_t \;=\; -\frac{\omega\mu}{\beta}\vec{e}_z \times \underline{\vec{H}}_t \;=\; -\frac{k}{\beta}Z_F\vec{e}_z \times \underline{\vec{H}}_t$$

If we introduce Z_{FH} the wave impedance of TE waves with

$$Z_{FH} \;=\; \frac{k}{\beta}Z_F \tag{2.17}$$

we get the equation for the transverse electric field its final form

$$\underline{\vec{E}}_t(x,y) \;=\; -Z_{FH}[\vec{e}_z \times \underline{\vec{H}}_t(x,y)] \tag{2.18}$$

To find the TE waves of a given cross section one has to solve eqution 2.15 under appropriate boundary conditions and with the help of equations 2.16 and 2.18 the tranverse magnetic, respectivily the electric fields are derminable.

2.1.3 TM waves

The TM or E waves have $\underline{H}_z \equiv 0$ but the axial electric field is not zero. These modes may be considered the dual of the TE modes in that the roles of the electrci and magnetic fields are interchanged. So in the following subsection we only will give the results of considerations that are similair to those given in the preceeding subsection.

First we have to find a solution for the partial differntial equation 2.19 of the axial electric field component $\underline{E}_z(x,y)$ and to fullfill the boundary conditions.

$$\vec{\nabla}^2\underline{E}_z(x,y) + k_c^2\underline{E}_z(x,y) \;=\; 0 \tag{2.19}$$

These will lead to the eigenvalues of the modes. Then the transverse electric field is given by

$$\underline{\vec{E}}_t(x,y) \;=\; -\frac{\beta}{k_c^2}\vec{\nabla}_t\underline{E}_z(x,y) \tag{2.20}$$

The transverse magentic field is then given by the following equation

$$\underline{\vec{H}}_t(x,y) \;=\; Y_{FE}[\vec{e}_z \times \underline{\vec{E}}_t(x,y)] \tag{2.21}$$

where we have introduced the wave addmittance of TM waves given by

$$Y_{FE} \;=\; \frac{1}{Z_{FE}} \;=\; \frac{k}{\beta}Y_F \tag{2.22}$$

2.2 Rectangular waveguide

The rectangular waveguide with a cross section as illustrated in Figure 2.3 is an example of a waveguide that will not support TEM waves. Consequently, it turns

Figure 2.3: Cross section of a rectangulare waveguide

out that unique voltage and current waves do not exist and the analysis of the waveguide properties has to be carried out as a field problem rather then as a distributed circuit problem. The types of waves that can be supported in a hollow empty wave guide are the TE and TM modes discussed in the previous section.

2.2.1 TE waves

For TE or H modes we have $E_z \equiv 0$ and all remaining field components can be determined from the axial magnetic field $H_z(x,y)$ which has to solve equtaion 2.15. Writen in a component notation this equations reads

$$\frac{\partial^2 H_z(x,y)}{\partial x^2} + \frac{\partial^2 H_z(x,y)}{\partial y^2} + k_c^2 H_z(x,y) = 0$$

If we assume $H_z(x,y)$ is writeable as a product of two independent function $X(x)$ and $Y(y)$ we are able to rewrite the above equation in the following form

$$\frac{1}{X(x)}\frac{\partial X(x)}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial Y(y)}{\partial y^2} + k_c^2 = 0$$

The first term is a function of x only, whereas the second term is a function of y only and k_c^2 is a constant. Hence the above equation can hold for all x and y values only if each term itself is constant. These leads to the seperation condition

$$k_c^2 = k_x^2 + k_y^2 \tag{2.23}$$

So we have to solve the following ordinary differential equations

$$\begin{aligned}\frac{d^2X(x)}{dx^2} + k_x^2 X(x) &= 0\\ \frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) &= 0\end{aligned}$$

It is well known that these ordinary differntial equations are solved from functions like sin(), cos(), exp() or any linear combination of these functions. To further specific the right function and the constants we have to consider the boundary conditions that have to be imposed on $H_(x,y)$. From equation 2.16 we not that the transverse magnetic field of a TE mode is given by the gradient of H_z. From 1.42 we known that no magnetic flux must enter the perfectly conducting walls of the rectangulare waveguide. Hence the approprated function for $H_z(x,y)$ has to fullfill the following boundary conditions.

$$\begin{aligned}\vec{e}_x\cdot\vec{H}_t(x=0,y) &= 0 \;\rightarrow\; \frac{\partial}{\partial x}H_z(x=0,y) = 0\\ \vec{e}_x\cdot\vec{H}_t(x=a,y) &= 0 \;\rightarrow\; \frac{\partial}{\partial x}H_z(x=a,y) = 0\\ \vec{e}_y\cdot\vec{H}_t(x,y=0) &= 0 \;\rightarrow\; \frac{\partial}{\partial y}H_z(x,y=0) = 0\\ \vec{e}_y\cdot\vec{H}_t(x,y=b) &= 0 \;\rightarrow\; \frac{\partial}{\partial y}H_z(x,y=b) = 0\end{aligned}$$

If we consider $X(x)$ to be $\cos(k_x x)$ then this function differentiated with respect to x would become the function $-\sin(k_x x)$, which fullfills automatically the boundary condition at $x = 0$. To fullfill the boundary condition at $x = a$ we have to assure

$$\sin(k_x a) = 0 \quad \text{which leads to} \quad k_x = \frac{n\pi}{a}$$

The same considerations hold true for the function $Y(y)$. Thus we find for the seperation constant k_y

$$k_y = \frac{m\pi}{b}$$

hence as appropriate solution for the $H_z(x,y)$ we have

$$H_z(x,y) = H_{n,m}\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b}) \qquad (2.24)$$

In this equation the constant $H_{n,m}$ is an arbitrary amplitude associated with the mode $\text{TE}_{n,m}$. As a result of the boundary conditions we find for the constant k_c

$$k_c(n,m) = \sqrt{(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2}$$

The phase propagation constant of each mode is then given by

$$\beta(n,m) = \sqrt{k_0^2 - k_c^2(n,m)}$$

For $k_0 > k_c(n,m)$, $\beta(n,m)$ is real and the mode will propagate. In the opposite case $\beta(n,m)$ will be imaginary and the mode will decay rapidly with the distance from the point at which it is exited. For this reason $k_c(n,m)$ is termed cutoff wave number. Directly connected with its value is the cutoff wavelength $\lambda_c(n,m)$

$$\lambda_c(n,m) = \frac{2\pi}{k_c(n,m)}$$

or in a slithly different form

$$\lambda_c(n,m) = \frac{2a}{\sqrt{n^2 + (\frac{ma}{b})^2}} \tag{2.25}$$

The decay of a mode is not associated with energy loss, but is a characteristic feature of these solutions. Such decaying or evansescent modes may be used to represent local diffraction or fringing fields that exist in the vicinity of coupling probes or obstacles in a waveguide. The frequency separating the propagation and non-propagation bands is designated the cutoff frequency $f_c(n,m)$

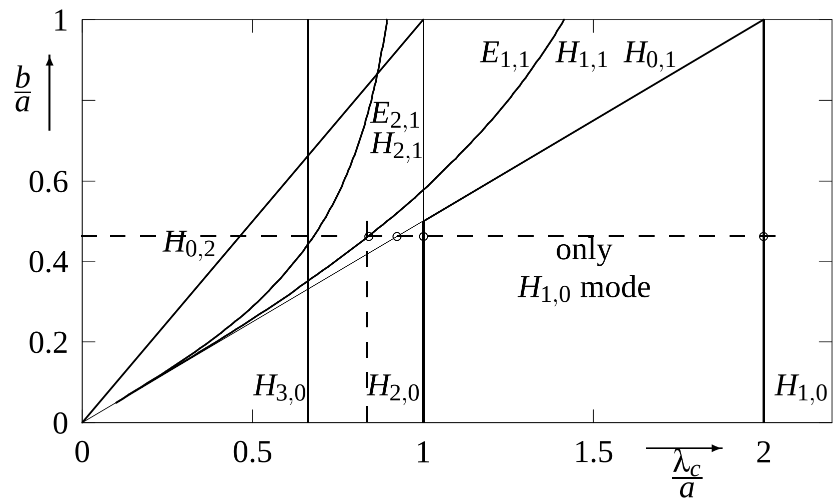

Figure 2.4: Mode chart

$$f_c(n,m) = \frac{c}{\lambda_c(n,m)} \tag{2.26}$$

To calculate all field components of the TE modes we have to use equation 2.16 and 2.18. In generall a rectangular waveguide does also support TM modes. Since the calculations are very similare table **??** gives in summary all field components of TE as well as TM modes, that may exist in a rectangulare waveguide.

	TE modes	TM modes
H_z	$\cos(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$	0
E_z	0	$\sin(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$
E_x	$Z_{H,nm}H_y$	$-j\frac{\beta_{nm}n\pi}{ak_{c,nm}^2}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$
E_y	$-Z_{H,nm}H_x$	$-j\frac{\beta_{nm}m\pi}{bk_{c,nm}^2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$
H_x	$j\frac{\beta_{nm}n\pi}{ak_{c,nm}^2}\sin(\frac{n\pi x}{a})\cos(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$	$-\frac{E_y}{Z_{E,nm}}$
H_y	$j\frac{\beta_{nm}m\pi}{bk_{c,nm}^2}\cos(\frac{n\pi x}{a})\sin(\frac{m\pi y}{b})\exp(-j\beta_{nm}z)$	$\frac{E_x}{Z_{E,nm}}$
$Z_{H,nm}$	$\frac{k_0}{\beta_{nm}}Z_0$	
$Z_{E,nm}$		$\frac{\beta_{nm}}{k_0}Z_0$
$k_{c,nm}$	$\sqrt{(\frac{n\pi}{a})^2+(\frac{m\pi}{b})^2}$	
β_{nm}	$\sqrt{k_0^2-k_{c,nm}^2}$	
$\lambda_{c,nm}$	$\frac{2ab}{\sqrt{n^2b^2+m^2a^2}}$	

Table 2.1: TE and TM field components of a rectangulare waveguide

Power

To calculate the power that a single mode transports, we have to evaluate the pointing vector with the help of equation 1.31. In the case of TE modes the electric field has only compoments in the transversal plane, while the magnetic field has components in all directions, hence we have to examine the following cross product

$$\begin{aligned}\underline{\vec{E}} \times \underline{\vec{H}}^* &= (\underline{E}_x \vec{e}_x + \underline{E}_y \vec{e}_y) \times (\underline{H}_x^* \vec{e}_x + \underline{H}_y^* \vec{e}_y + \underline{H}_z^* \vec{e}_z) \\ \underline{\vec{E}} \times \underline{\vec{H}}^* &= (\underline{E}_x \underline{H}_y^* - \underline{E}_y \underline{H}_x^*) \vec{e}_z - \underline{E}_x \underline{H}_z^* \vec{e}_y + \underline{E}_y \underline{H}_z^* \vec{e}_x\end{aligned}$$

Only the first term of the last equation will contribute to a power transport in z-direction. To compute the total power we have to integrate this term over the cross section of the waveguide. So we find for the total power $P_{n,m}$ transported by the mode $\mathrm{TE}_{n,m}$

$$\begin{aligned}P_{n,m} &= \frac{1}{2}\int_0^a \int_0^b \mathbf{Re}(\underline{E}_x \underline{H}_y^* - \underline{E}_y \underline{H}_x^*) dxdy \\ &= \frac{1}{2}\int_0^a \int_0^b (|\underline{H}_y|^2 + |H_x|^2) dxdy\end{aligned}$$

As result of this integration one finds

$$P_{n,m} = \frac{1}{2} Z_{FH}(n,m) |H_{n,m}|^2 \frac{1/2ab}{\delta_n \delta_m} \left[(\frac{n\pi}{a})^2 + (\frac{m\pi}{b})^2 \right] \tag{2.27}$$

with δ_n beeing the so called Neumann factor which has the following properties

$$\delta_m = \begin{cases} 2 & \text{for} \quad n \neq 0 \text{ and } m \neq 0 \\ 1 & \text{for} \quad m = 0 \end{cases}$$

2.2.2 TE$_{10}$ mode

From Figure 2.4 it is clear that the TE_{10} mode is for $b \leq a/2$ the mode with the lowest cutoff frequency. It is the most commonly used mode, that the reason, why we examine this mode in more detail. Table 2.2 gives a summary of important technical used wave guides. Instead of starting with the H_z component, we rewrite the field components of the $H_{1,0}$ mode by starting with the electric field

$$\begin{aligned}\underline{E}_y(x,z) &= \underline{E}_{y0} \sin(\frac{\pi}{a}x) \exp(-jk_z z) \\ \underline{H}_x(x,z) &= -\frac{1}{Z_{FH}} \underline{E}_y(x,z) \\ \underline{H}_z(x,z) &= j\frac{k_x}{\omega\mu} \underline{E}_{y0} \cos(\frac{\pi}{a}x) \exp(-jk_z z)\end{aligned} \tag{2.28}$$

Designation	Frequency range in GHz	Dimensions in mm×mm
R32	2,60 - 3,95	72,14×34,04
R48	3,94 - 5,99	47,55×22,15
R70	5,38 - 8,17	34,85×15,80
R100	8,20 - 12,5	22,86×10,16
R140	11,9 - 18,0	15,80×7,90
R220	17,6 - 26,7	10,67×4,32

Table 2.2: Important waveguides

$$\lambda_z = \frac{\lambda}{\sqrt{1-(\frac{\lambda}{2a})^2}} \quad (2.29)$$

$$Z_{FH} = \frac{Z_F}{\sqrt{1-(\frac{\lambda}{2a})^2}} \quad (2.30)$$

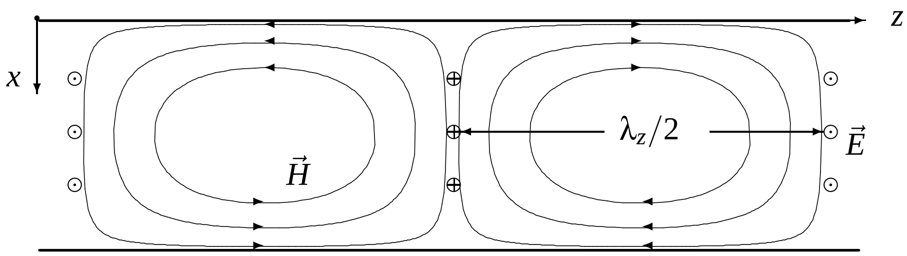

Figure 2.5: Fielddistribution of the $H_{1,0}$ mode

Phase- and groupvelocicty

$$v_{Ph} = \frac{\omega}{k_z} = \frac{f\lambda}{\sqrt{1-(\frac{\lambda}{\lambda_c})^2}} = \frac{c}{\sqrt{1-(\frac{\lambda}{\lambda_c})^2}} \quad (2.31)$$

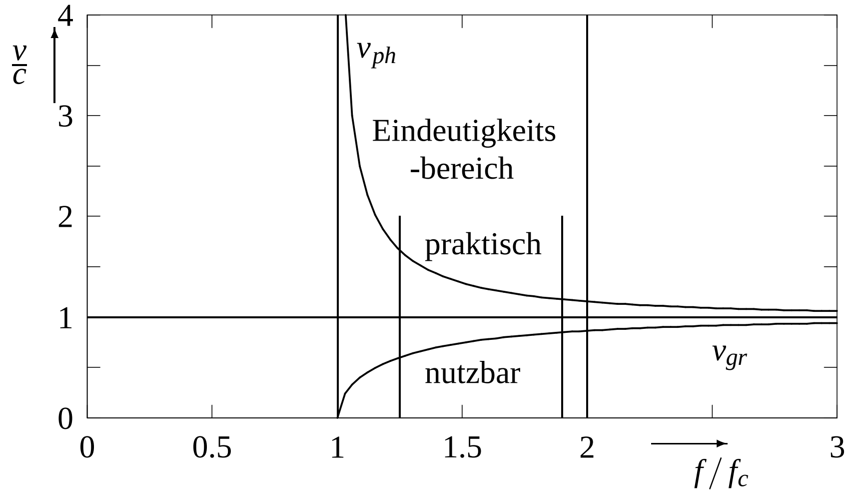

Figure 2.6: Phase- and groupvelocity of $H_{1,0}$ mode

Attenuation

$$\vec{J}_{Ol} = \vec{e}_x \times \underline{\vec{H}}(x=0) = -j\frac{\lambda_z/2}{aZ_{FH}}\underline{E}_{y0}\vec{e}_y$$

$$\vec{J}_{Or} = -\vec{e}_x \times \underline{\vec{H}}(x=a) = j\frac{\lambda_z/2}{aZ_{FH}}\underline{E}_{y0}\vec{e}_y$$

$$\vec{J}_{Oo} = -\vec{e}_y \times \underline{\vec{H}}(x) = -\frac{\underline{E}_{yo}}{Z_{FH}}\sin(\frac{\pi}{a}x)\vec{e}_z - j\frac{\lambda_z/2}{aZ_{FH}}\underline{E}_{yo}\cos(\frac{\pi}{a}x)\vec{e}_x$$

$$\vec{J}_{Ou} = \vec{e}_y \times \underline{\vec{H}}(x) = \frac{\underline{E}_{yo}}{Z_{FH}}\sin(\frac{\pi}{a}x)\vec{e}_z + j\frac{\lambda_z/2}{aZ_{FH}}\underline{E}_{yo}\cos(\frac{\pi}{a}x)\vec{e}_x$$

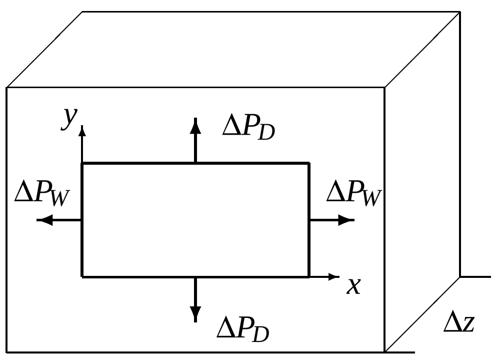

Figure 2.7: Rectangular waveguide of length Δz

$$\Delta P_W = \frac{1}{2}R_O \int_0^b \left(|\vec{J}_{Or}|^2 + |\vec{J}_{Ol}|^2\right) dy\Delta z = \frac{aR_O}{2Z_{FH}^2}(\frac{\lambda_z/2}{a})^2 2\frac{b}{a}|\underline{E}_{y0}|^2\Delta z$$

$$\Delta P_D = \frac{1}{2}R_O \int_0^a \left(|\vec{J}_{Oo}|^2 + |\vec{J}_{Ou}|^2\right) dx\Delta z = \frac{aR_O}{2Z_{FH}^2}(\frac{\lambda_z/2}{a})^2|\underline{E}_{y0}|^2\Delta z + \frac{aR_O}{2Z_{FH}^2}|\underline{E}_{y0}|^2\Delta z$$

$$\alpha = \frac{R_O}{Z_F b\sqrt{1-(\frac{f_c}{f})^2}}\left(1+2\frac{b}{a}(\frac{f_c}{f})^2\right) \tag{2.32}$$

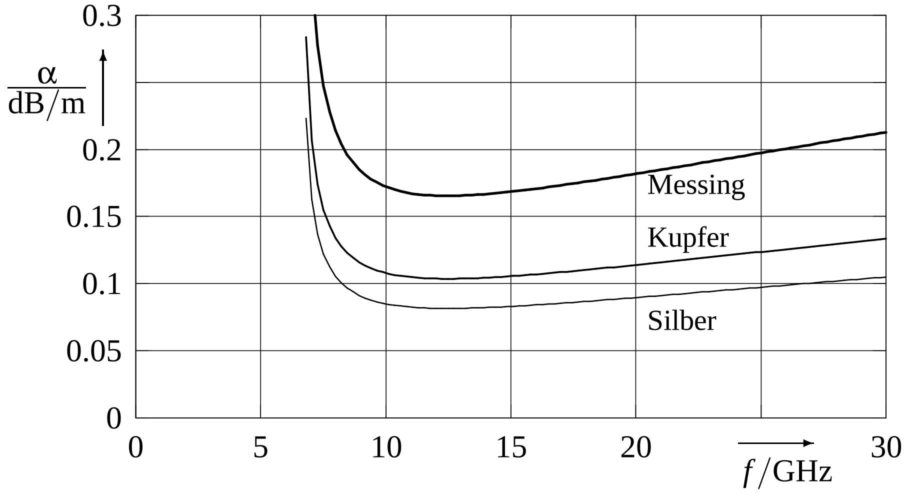

Figure 2.8: Attenuation of a R100 waveguide for different metalls

2.3 Circular waveguide

Figure 2.9 shows the cross section of a circulare wave guide. To describe its geometry cylindrical coordinates are most appropriate for the analysis to be carried out. In the preceeding section we allready discussed that in an arbitrary hollow cross section allways TE and TM modes will exist. To start the calculation we first have to study the nabla operator in a cylindrical coordinate system. Its transvers components are given by

$$\vec{\nabla}_t = \vec{e}_\rho \frac{\partial}{\partial\rho} + \vec{e}_\varphi \frac{\partial}{\rho\partial\varphi}$$

Figure 2.9: Cross section of a circulare waveguide

Form equation 2.1 we known that we have to consider the square of the nabla operator. Compared to the cartessian case, in a cylindrical coordinate system its square is some what more complicated to calculate since one has to remeember that the unit vectors $\vec{e}_\rho$ and $\vec{e}_\varphi$ itself are depending of the φ-coordinate of the system. If we rememeber this we get

$$\vec{\nabla}_t^2 = \frac{\partial^2}{\partial\rho^2} + \frac{\partial}{\rho\partial\rho} + \frac{\partial^2}{\rho^2\partial\varphi^2} \tag{2.33}$$

That means that the z component of the electric field has to fullfill the following differential equation

$$\frac{\partial^2\underline{E}_z}{\partial\rho^2} + \frac{\partial\underline{E}_z}{\rho\partial\rho} + \frac{\partial^2\underline{E}_z}{\rho^2\partial\varphi^2} + k_c^2\underline{E}_z = 0$$

The method of seperating the variables may here also be applied to end up with ordinary differential equations. If we suppose $\underline{E}_z$ to have the following form

$$\underline{E}_z(\rho,\varphi) = E_{z0}\, f(\rho)\, g(\varphi)$$

we insert this into the partial differential equation and divide it by the functions itself we get

$$\frac{1}{f(\rho)}\left[\frac{\partial^2 f(\rho)}{\partial\rho^2} + \frac{\partial f(\rho)}{\rho\partial\rho}\right] + \frac{1}{g(\varphi)}\frac{\partial^2 g(\varphi)}{\rho^2\partial\varphi^2} + k_c^2 = 0$$

multiplication with ρ^2 yields

$$\frac{\rho^2}{f(\rho)}\left[\frac{\partial^2 f(\rho)}{\partial\rho^2} + \frac{\partial f(\rho)}{\rho\partial\rho}\right] + \rho^2 k_c^2 = -\frac{1}{g(\varphi)}\frac{\partial^2 g(\varphi)}{\partial\varphi^2}$$

The left hand side is a function of ρ only, whereas the right hand side depends only on φ. Therefore this equation can hold for all values of the variables only if both sides are equal to a constant, say ν^2. Hence we have

$$\frac{\rho^2}{f(\rho)}\left[\frac{\partial^2 f(\rho)}{\partial\rho^2}+\frac{\partial f(\rho)}{\rho\partial\rho}\right]+\rho^2 k_c^2 = \nu^2$$

and

$$-\frac{1}{g(\varphi)}\frac{\partial^2 g(\varphi)}{\partial\varphi^2} = \nu^2$$

So we have to solve the following ordinary differential equations

$$\frac{d^2 f(\rho)}{d\rho^2}+\frac{d f(\rho)}{\rho d\rho}+\left[\rho^2 k_c^2-\frac{\nu^2}{\rho^2}\right] f(\rho) = 0$$
$$\frac{d^2 g(\varphi)}{d\varphi^2}+\nu^2 g(\varphi) = 0$$

Of course of the circular structure the field inside the waveguide must be periodic in φ with period 2π. Hence the general solution for the function $g(\varphi)$ would be a weighted sum of $\cos(n\varphi)$ and $\sin(n\varphi)$, n beeing an integer. But since there is essentialy no difference between the two function we only choose the sin function. The term with the cos-function would then belong to a denegerated mode with perpenddicular polarization. The differential equation for the function $f(\rho)$ has also two independent solutions. The solutions are the bessel functions of first $J_n(k_c\rho)$ and second kind $Y_n(k_c\rho)$. Since the function Y_n becomes infinite as ρ approches zero, the only physically acceptable solutions are the fessel functions of first kind. So the generallel solution for a TM mode reads

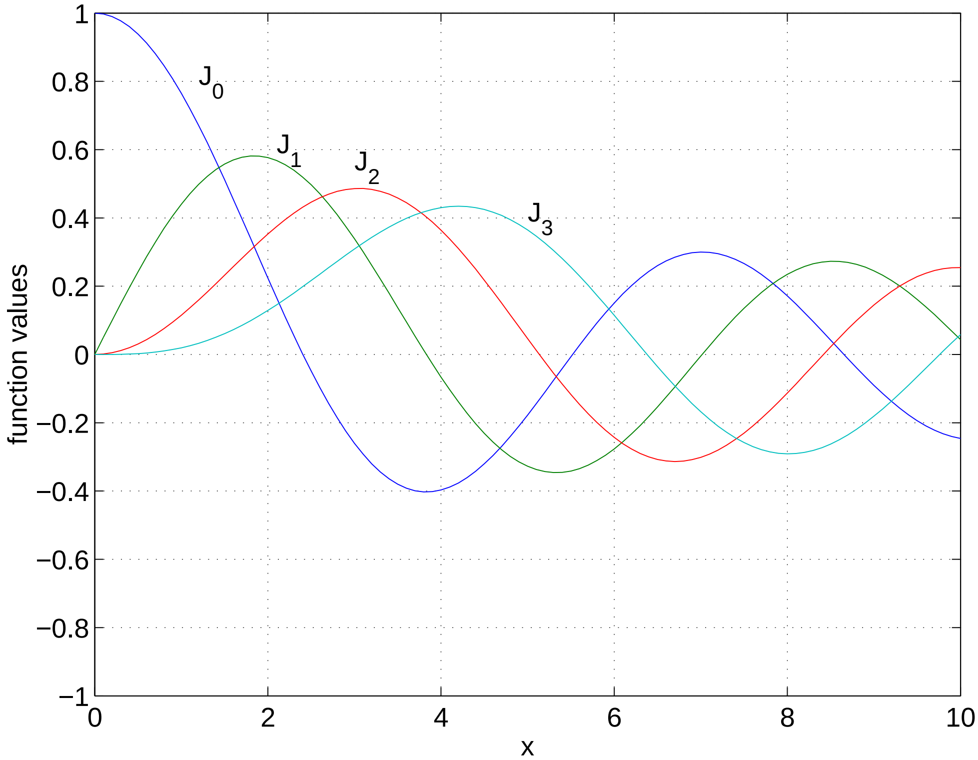

Figure 2.10: Bessel functions of first kind

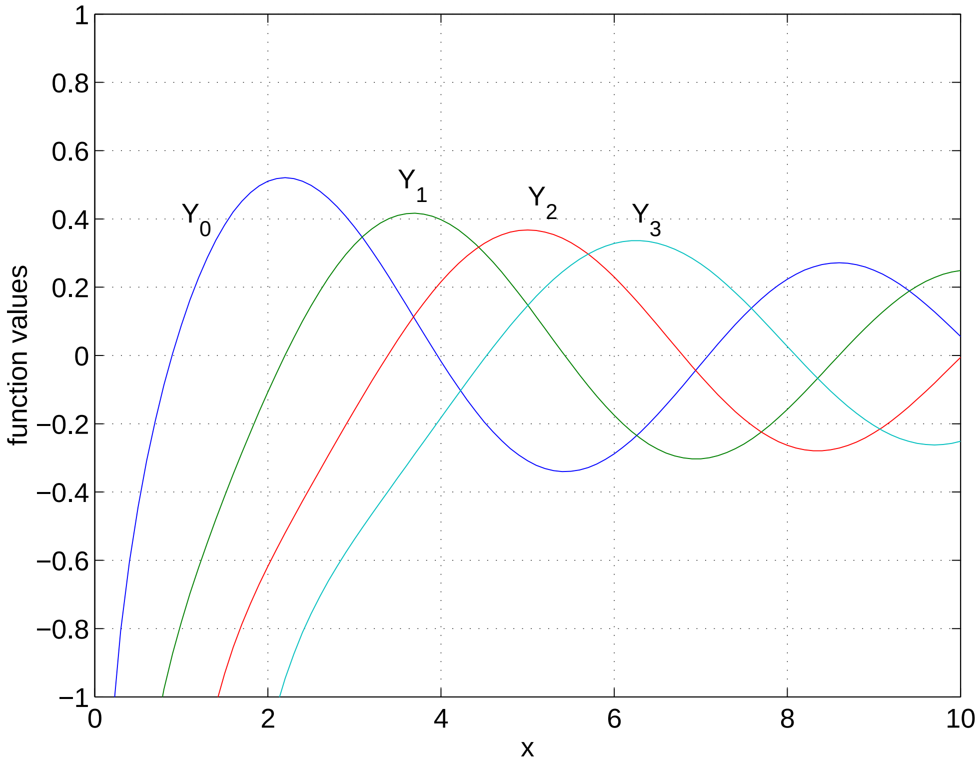

Figure 2.11: Bessel functions of second kind

$$\underline{E}_z(\rho, \varphi) \;=\; E_{z0}\sin(n\varphi)J_n(k_c\rho)$$

Since E_z must vanish at $\rho = a$, it is necessary to choose $k_c a$ in such a manner that $J_n(k_c a) = 0$. If the m'th root of the equation $J_n(x) = 0$ is designated $p_{n,m}$. the allowed eigenvalues of k_c are

$$k_c(n,m) \;=\; \frac{p_{n,m}}{a} \tag{2.34}$$

The propagation constant $\beta(n,m)$ of the $TM_(n,m)$ mode is given by

$$\beta(n,m) \;=\; \sqrt{k^2 - (\frac{p_{n,m}}{a})^2} \tag{2.35}$$

Table 2.3 shows the field components and the modes of the circulare waveguide.

	TE modes	TM modes
H_z	$J_n(\frac{p'_{nm}\rho}{a})\cos(n\phi)\exp(-j\beta_{nm}z)$	0
E_z	0	$J_n(\frac{p_{nm}\rho}{a})\cos(n\phi)\exp(-j\beta_{nm}z)$
H_ρ	$-j\frac{\beta_{nm}p'_{nm}}{ak^2_{c,nm}}J'_n(\frac{p'_{nm}\rho}{a})\cos(n\phi)\exp(-j\beta_{nm}z)$	$-\frac{E_\phi}{Z_{E,nm}}$
H_ϕ	$j\frac{n\beta_{nm}}{\rho k^2_{c,nm}}J_n(\frac{p'_{nm}\rho}{a})\sin(n\phi)\exp(-j\beta_{nm}z)$	$\frac{E_\rho}{Z_{E,nm}}$
E_ρ	$Z_{H,nm}H_\phi$	$-j\frac{\beta_{nm}p_{nm}}{ak^2_{c,nm}}J'_n(\frac{p_{nm}\rho}{a})\cos(n\phi)\exp(-j\beta_{nm}z)$
E_ϕ	$-Z_{H,nm}H_\rho$	$j\frac{n\beta_{nm}}{\rho k^2_{c,nm}}J_n(\frac{p_{nm}\rho}{a})\sin(n\phi)\exp(-j\beta_{nm}z)$
$Z_{H,nm}$	$\frac{k_0}{\beta_{nm}}Z_0$	
$Z_{E,nm}$		$\frac{\beta_{nm}}{k_0}Z_0$
β_{nm}	$\sqrt{k_0^2-(\frac{p'_{nm}}{a})^2}$	$\sqrt{k_0^2-(\frac{p_{nm}}{a})^2}$
$k_{c,nm}$	$\frac{p'_{nm}}{a}$	$\frac{p_{nm}}{a}$
$\lambda_{c,nm}$	$\frac{2\pi a}{p'_{nm}}$	$\frac{2\pi a}{p_{nm}}$

Table 2.3: Field components of circular waveguide modes

TE			
n	p'_{n1}	p'_{n2}	p'_{n3}
0	3,832	7,016	10,174
1	1,841	5,331	8,536
2	3,054	6,706	9,970

TM			
n	p_{n1}	p_{n2}	p_{n3}
0	2,405	5,520	8,654
1	3,832	7,016	10,174
2	5,135	8,417	11,620

Table 2.4: Values of $p'_{n,m}$ and $p_{n,m}$ for TE and TM modes

2.4 Optical waveguides

2.4.1 Dielectric slab waveguide

One of the simplest dielectric waveguides is the symmetric slab waveguide shown in Figure 2.12. It is formed by a dieletric sheet with refractive index n_1 sur-

Figure 2.12: Cross section of a dielectric slab waveguide

rounded symmetrically by second dielectric having refractive index $n_2 < n_1$. We will search for TE modes propagating in this waveguide. Since the fields are supposed to have no variations in the y-direction the transvers nabla operator reduces to

$$\vec{\nabla}_t = \vec{e}_x \frac{\partial}{\partial x} \quad \rightarrow \quad \vec{\nabla}_t^2 = \frac{\partial^2}{\partial x^2}$$

A single wave is described by a propagation constant β for both regions. This will lead to different k_c values depending on the region we will consider. In the region of refractive index n_1 we assume k_c to be real and denote it by k_d,

$$\frac{d^2\underline{H}_z(x)}{dx^2} + k_d^2 \underline{H}_z(x) = 0 \quad \text{for} \quad -d \leq x \leq d$$

in the region of refractive index n_2 we assume k_c to be imaginary and we denote by $j\alpha$:

$$\frac{d^2\underline{H}_z(x)}{dx^2} - \alpha^2 \underline{H}_z(x) = 0 \quad \text{for} \quad x > d$$

The first differential equation is solved by a sin or cos-function

$$H_z(x) = H_{z1} \sin(k_d x) \quad \text{or} \quad H_z(x) = H_{z1} \cos(k_d x)$$

while the last differential equation is solved by a exponential function

$$H_z(x) = H_{z2} \exp(-\alpha x) \quad \text{for} \quad x > d$$

At the dielectric boundary $x = d$ the tangential electric and magnetic fields must be continous for all values of z. This requires that the propagation constant β must be same in both regions which results in the following equation

$$\begin{aligned} \beta^2 &= k_1^2 - k_d^2 = k_2^2 + \alpha^2 \\ k_1^2 - k_2^2 &= k_d^2 + \alpha^2 \\ k_0^2(n_1^2 - n_2^2) &= k_d^2 + \alpha^2 \end{aligned} \tag{2.36}$$

The continuity of the magnetic field at the inferface results in two equations depending on the mode we consider

$$H_{z1}\sin(k_d d) = H_{z2}\exp(-\alpha d) \qquad \text{or} \qquad H_{z1}\cos(k_d d) = H_{z2}\exp(-\alpha d) \tag{2.37}$$

One further relation is necessary in order to determine the quantities H_{z2}/H_{z1}, k_d and α. This relation is obtain from the requirement that the electric field component E_y also has to be continuous at $x = d$. For E_y we find from equation ??

$$\underline{\vec{E}}_t = -\frac{\omega\mu_0}{\beta}\vec{e}_z \times \underline{\vec{H}}_t \quad \text{with} \quad \underline{\vec{H}}_t = -j\frac{\beta}{k_c^2}\vec{\nabla}_t H_z$$

$$\underline{\vec{E}}_t = jZ_0\frac{k_0^2}{k_c^2}\vec{e}_z \times [\vec{\nabla} H_z] \tag{2.38}$$

We have to distinguish the two regions. In the first region we have $k_c^2 = k_d^2$ and we find

$$\underline{E}_y = jZ_0\frac{k_0}{k_d}H_{z1}\cos(k_d d) \quad \text{or} \quad \underline{E}_y = -jZ_0\frac{k_0}{k_d}H_{z1}\sin(k_d d)$$

in the second we have $k_c^2 = -\alpha^2$ and we find for the electric field

$$\underline{E}_y = jZ_0\frac{k_0}{\alpha}H_{z2}\exp(-\alpha x)$$

The continuity of the electric fields at the interface results in

$$\frac{1}{k_d}H_{z1}\cos(k_d d) = \frac{1}{\alpha}H_{z2}\exp(-\alpha d) \quad \text{or} \quad -\frac{1}{k_d}H_{z1}\sin(k_d d) = \frac{1}{\alpha}H_{z2}\exp(-\alpha d) \tag{2.39}$$

If we divide equation 2.37 by equation 2.39 we find:

$$k_d d\tan(k_d d) = \alpha d \quad \text{or} \quad -k_d d\cot(k_d d) = \alpha d \tag{2.40}$$

With the help of equation 2.36 we introduce the so called V-number of an dielectric slab waveguide:

$$V^2 = k_0^2 d^2\left(n_1^2 - n_2^2\right) \tag{2.41}$$

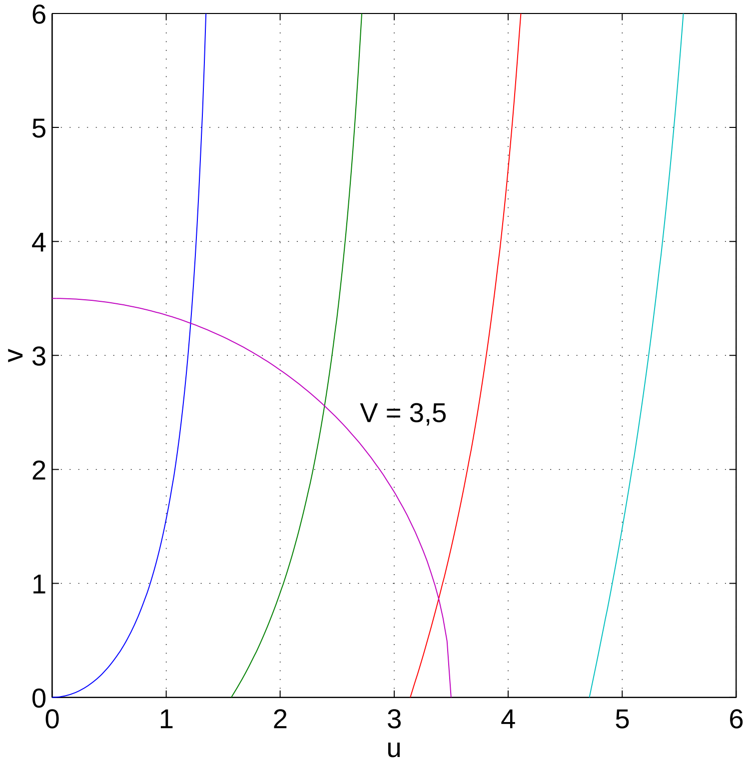

Figure 2.13: Graphic to determine the propagation constant of a slab wavequide

and the normalized values $u = k_d d$ and $v = \alpha d$. Thus equation 2.36 can be rewritten in the following form:

$$V^2 = u^2 + v^2 \tag{2.42}$$

Figure 2.13 represents equation 2.40 and 2.42 in a graphical form. Since both equations have to be satisfied only the points of intersection between the circle and the functions 2.42 yield possible values of u and v for a given dielectric slab wavequide described by the V-Parameter.