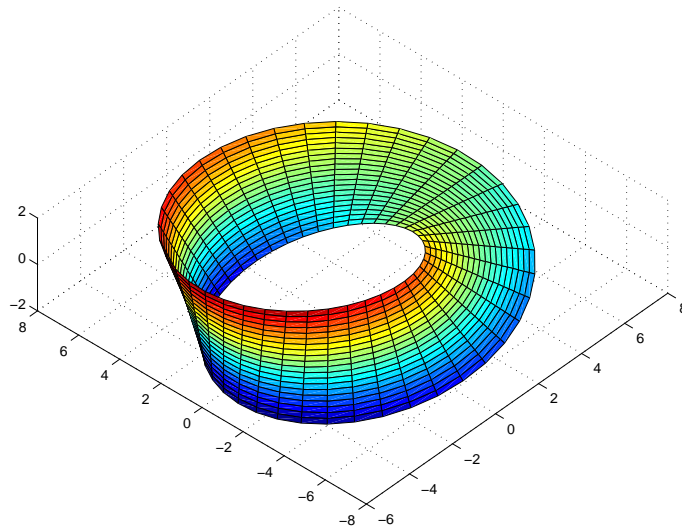


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Vector Analysis

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Preface

This text is intended for a one-semester course in the calculus of functions of several variables and vector analysis with special emphasis on applications in electromagnetic field theory. Since the subject of this course is a well established domain in mathematics, there exists a huge amount of textbooks as for example [1], [2], [3] and [4] to mention only some of them. To get really familiar with the field of vector analysis the student is encouraged to extend his studies beyond this short text.

Chapter 1

The geometry of euclidean space

1.1 Vectors in three-dimensional space

To illustrate the vector concept we will first consider a vector given in a plane and represent it with the help of cartesian coordinates as shown in Figure 1.1 The vector

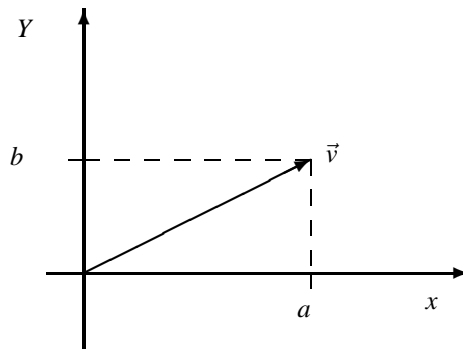


Figure 1.1: Vector in a plane, represented in cartesian coordinates

\vec{v} is denoted by a set of two ordered numbers with a being the x- and b being the y-component of the vector.

$$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$$

As illustrated in Figure 1.2 a vector or a point in three dimensional space may be represented as a ordered triple of real numbers.

$$\vec{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

In this order a stands for the x-, b for the y- and c for the z-component of the vector. To distinct the different dimensions of a real space, we will use the following notations:

- One dimensional space, a line, is denoted by \mathbf{R}^1

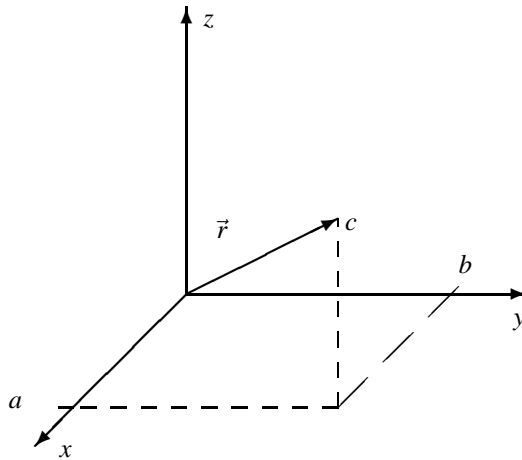


Figure 1.2: Vector in a three dimensional space, represented in cartesian coordinates

- Two dimensional space, a plane, is denoted by \mathbf{R}^2
- Three dimensional space, is denoted by \mathbf{R}^3

The extension of the notation to spaces with arbitrary dimensions is straight forward.

1.1.1 Simple vector operations

Considered a vector as a mathematical object different operations on this objects may be defined. These definitions are motivated to give meaningful results when applied two physical objects like, velocities or forces, which are normally represented by vectors.

Addition of vectors

Suppose there are defined two vectors \vec{a} and \vec{b} and they are given in their cartesian representation.

$$\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$$

Then the result \vec{c} of an addition of the two vectors is defined by

$$\vec{c} = \vec{a} + \vec{b} \quad \Rightarrow \quad \vec{c} = \begin{pmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{pmatrix}$$

In Figure 1.3 the addition of two vectors is illustrated. Imagine two vectors describing the movement of e.g. an object in two dimensional space, the total movement described by the vector \vec{c} is given by the two subsequent displacements described by the vectors \vec{a} and \vec{b} independent of the fact which of them is done first. This fact is termed commutativity of the operation.

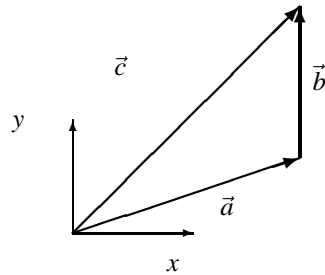


Figure 1.3: Addition of vectors

Multiplication by a scalar

There are a number of product operations that can be defined for vectors. The simplest operation is the multiplication of a vector \vec{v} by a real number $\alpha \in \mathbf{R}^1$. This operation is denoted as scalar multiplication and the definition reads

$$\vec{c} = \alpha \vec{v} \Rightarrow \vec{c} = \begin{pmatrix} \alpha v_x \\ \alpha v_y \\ \alpha v_z \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$$

To illustrate the geometric meaning of the scalar multiplication Figure 1.4 shows a vector \vec{v} and the result vector \vec{c} for the special case of $\alpha > 1$. In this case the resulting vector \vec{c} has the same direction but is longer than the original vector. In the case $\alpha < 0$ the resulting vector would have the opposite direction.

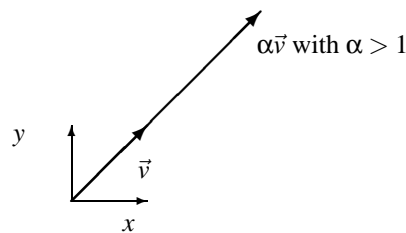


Figure 1.4: Multiplication of a vector by a scalar

Subtraction of vectors

With the help of the last operation the subtraction of vectors can be reduced to a scalar multiplication with $\alpha = -1$ and a subsequent addition of these vectors.

$$\vec{c} = \vec{a} - \vec{b} = \vec{a} + (-1 \cdot \vec{b}) = \vec{a} + \vec{b}_n \quad \text{with} \quad \vec{b}_n = \begin{pmatrix} -b_x \\ -b_y \\ -b_z \end{pmatrix}$$

Figure 1.5 shows an example for the subtraction of two dimensional vectors.

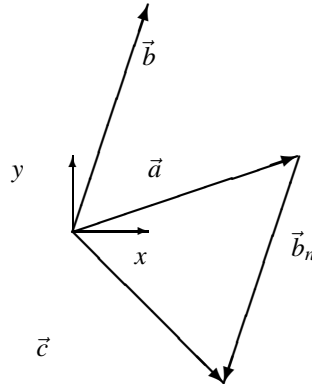


Figure 1.5: Subtraction of vectors

Standard basis vectors in \mathbf{R}^3

We define the following standard basis vectors

$$\vec{u}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{u}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{u}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (1.1)$$

With the help of vector addition and scalar multiplication, we find that any arbitrary vector $\vec{v} = (v_x, v_y, v_z)$ may be expressed by a weighted sum of the standard basis vectors.

$$\vec{v} = v_x \vec{u}_x + v_y \vec{u}_y + v_z \vec{u}_z$$

1.1.2 The inner product

The inner product of two vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$, sometimes also denoted as **scalar product** or **dot product** is defined by an operation on its components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \quad (1.2)$$

With the help of the definition of the dot product one can show that the following relations hold true

$$\begin{aligned} \vec{a} \cdot \vec{a} &\geq 0 \\ \vec{a} \cdot \vec{a} &= 0 \quad \text{if and only if} \quad \vec{a} \equiv 0 \\ (\alpha \vec{a}) \cdot \vec{b} &= \alpha (\vec{a} \cdot \vec{b}) \\ \vec{a} \cdot (\beta \vec{b}) &= \beta (\vec{a} \cdot \vec{b}) \\ \vec{a} \cdot \vec{b} &= \vec{b} \cdot \vec{a} \end{aligned} \quad (1.3)$$

The norm or length of a vector is defined by

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

A unit vector \vec{u}_a is a vector with norm unity. It can always be constructed from a vector \vec{a} with non zero length.

$$\vec{u}_a = \frac{1}{|\vec{a}|}\vec{a}$$

We will now show that the following relation for the inner product holds also true

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) \quad (1.4)$$

For \vec{a} and \vec{b} being to arbitrary vectors in \mathbf{R}^3 and θ being the angle between the two vectors with $0 \leq \theta \leq \pi$. To proof this relation we apply the law of cosines from trigonometry to the triangle spanned by the two vectors \vec{a} , \vec{b} and shown in Figure 1.6.

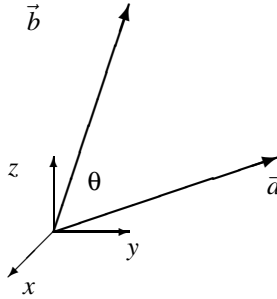


Figure 1.6: Triangle spanned by two vectors

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\theta)$$

Since $|\vec{b} - \vec{a}|^2 = (\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a})$ we can rewrite the above equation as:

$$(\vec{b} - \vec{a}) \cdot (\vec{b} - \vec{a}) = \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2\vec{b} \cdot \vec{a}$$

Comparing the above results proofs equation 1.4. Since the absolute value of the cosine function is always lower equal one the so called **Cauchy-Schwarz Inequality** follows directly from equation 1.4.

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

1.1.3 The cross product

In the last section we discussed the dot product of two vectors which results in a scalar value. In this section we will define the cross product which is some times also called the **vector product**, since it produces a vector as its result. To illustrate how one has to construct the result vector consider Figure 1.7. There are shown two arbitrary vectors \vec{a} and \vec{b} , which define a plane in the three dimensional space. The direction of the vector to be constructed is normal to the plane spanned by the two vectors, whereas its absolute value is equal to the area of the spanned parallelogram. Normally there exist always two vectors that are normal to a plane. In the cross product the normal direction \vec{u}_n has to be chosen, which gives the direction of a right turning screw, if the vector

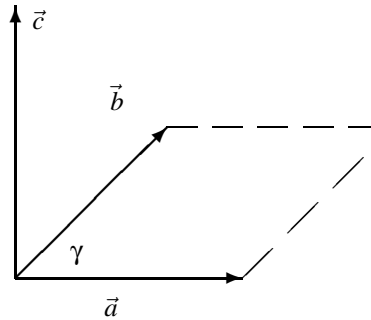


Figure 1.7: Geometry to define the result vector of the cross product

\vec{a} is turned into the vector \vec{b} using the smallest angle γ . This results in the following definition

$$\vec{c} = |\vec{a}||\vec{b}|\sin(\gamma)\vec{u}_n$$

As a consequence of this definition, the following equation holds true

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

and hence we have

$$\vec{a} \times \vec{a} = -\vec{a} \times \vec{a} \equiv 0$$

With the help of the above definition, one can also easily proof the following algebraic properties of the cross product:

$$\vec{a} \times (\alpha\vec{b} + \beta\vec{c}) = \alpha(\vec{a} \times \vec{b}) + \beta(\vec{a} \times \vec{c})$$

$$(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha(\vec{a} \times \vec{c}) + \beta(\vec{b} \times \vec{c})$$

For the standard base vectors defined in 1.1 we find the following results for their cross products

$$\begin{aligned}\vec{u}_x \times \vec{u}_y &= \vec{u}_z \\ \vec{u}_y \times \vec{u}_z &= \vec{u}_x \\ \vec{u}_z \times \vec{u}_x &= \vec{u}_y\end{aligned}$$

One can also determine the cross product of two vectors by symbolically equating the value of the following determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{u}_x & \vec{u}_y & \vec{u}_z \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} =$$

$$\vec{u}_x(a_y b_z - a_z b_y) - \vec{u}_y(a_x b_z - a_z b_x) + \vec{u}_z(a_x b_y - a_y b_x)$$

where we have done a determinant expansion by minors over the first row.

1.2 n-Dimensional Euclidian space

In the previous section we defined vectors and operations on these vectors in a three dimensional euclidean space. To generalize the concept of euclidean space to an arbitrary order of dimensions we define vectors in this space by a set of n ordered real numbers.

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

We launch our study of euclidean n-space by introducing several algebraic operations analogous to those allready defined in \mathbf{R}^3 . We define the addition of two vectors by the addition of their components.

$$\vec{c} = \vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \cdot \\ \cdot \\ a_n + b_n \end{pmatrix}$$

In the same way we define the multiplication by a scalar.

$$\vec{c} = \alpha \vec{a} = \begin{pmatrix} \alpha a_1 \\ \alpha a_2 \\ \cdot \\ \cdot \\ \alpha a_n \end{pmatrix}$$

One can also easily define a set of standard basis vectors in \mathbf{R}^n .

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \quad \dots \quad \vec{u}_n = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

Any vector \vec{a} can be written as weighted sum of the standard basis vectors.

$$\vec{a} = a_1 \vec{u}_1 + a_2 \vec{u}_2 + \dots + a_n \vec{u}_n$$

Since the dot or inner product in 3-dimensional space (1.2) was allready defined using the components of the vectors the generalization of this operation to n-dimensional space is straight forward.

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

With the help of the dot product, one can easily extend the concept of the norm or length of a vector to n-dimensional space.

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \dots + a_n^2}$$

As a consequence the cosine of the angle θ between two vectors in n -dimensional space is given by:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \quad (1.5)$$

Since there is really no difference considering the dot product in 3- or n -dimensional space the equations 1.3 hold also true for vectors in spaces of arbitrary dimensions. We remark that there is **no cross product** defined on \mathbf{R}^n except for $n = 3$. It is only the dot product that generalizes to n -dimensional space.

1.3 Linear mapping in Euclidean space

In order to describe a linear mapping of arbitrary vectors \vec{x} being elements of \mathbf{R}^n onto a space \mathbf{R}^m with elements \vec{y} we define a matrix as a 2-dimensional entity of $m \times n$ ordered real numbers, whereas m counts the number of rows and n counts the number of columns of the matrix.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \vdots & \vdots \cdots \vdots \\ a_{m1} & a_{m2} \cdots a_{mn} \end{pmatrix}$$

The linear mapping $\vec{x} \mapsto \vec{y}$ is then described by the following operation on the component values of the vector \vec{x} .

$$y_i = \sum_{k=1}^n a_{ik} x_k \quad \text{with } i \in (1 \cdots m)$$

One speaks of multiplying the vector \vec{x} with the matrix \mathbf{A} and uses the following short term notation.

$$\vec{y} = \mathbf{A}\vec{x}$$

It is natural to define the addition and the subtraction of two matrices of same dimension $m \times n$ by the addition and subtraction of the single components as well as the multiplication of a matrix by a scalar. As consequence of the above definitions the equations of linearity hold true.

$$\begin{aligned} \mathbf{A}(\vec{x} + \vec{y}) &= \mathbf{A}\vec{x} + \mathbf{A}\vec{y} \\ \mathbf{A}(\alpha\vec{x}) &= \alpha(\mathbf{A}\vec{x}) \end{aligned}$$

Chapter 2

Differentiation

In this chapter we will extend the principles of differential calculus for functions of one variable to functions of several variables, restricting ourself first to so called **real-valued functions**. In ordinary differential calculus only functions of one variable are considered, which is normally denoted by x , for example $f(x) = x^2$. From a geometrical point of view, this function maps x being an element of \mathbf{R}^1 onto $f(x)$ which is also an element of \mathbf{R}^1 or in a mathematical short term notation we have

$$f(x) = x^2 \quad : \quad \mathbf{R}^1 \mapsto \mathbf{R}^1$$

In the following chapter we will consider functions that describe a mapping e.g. from 2-dimensional space $(x, y) \in \mathbf{R}^2$ or 3-dimensional space $(x, y, z) \in \mathbf{R}^3$ onto a real value being an element of \mathbf{R}^1 . As variables we consider the ordered set (x, y) or (x, y, z) describing an arbitrary point in a cartesian coordinate system.

$$\begin{aligned} f(x, y) & : \quad \mathbf{R}^2 \mapsto \mathbf{R}^1 \\ f(x, y, z) & : \quad \mathbf{R}^3 \mapsto \mathbf{R}^1 \end{aligned}$$

As a first example of a real-valued function we will consider the potential function of a charge Q located in the center of a cartesian coordinate system.

$$\phi(x, y, z) = \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + z^2}} \quad \text{with} \quad \epsilon_0 = 1,256 \cdot 10^{-12} \frac{Vs}{Am} \quad (2.1)$$

From a mathematical point of view we have a function that maps roughly speaking \mathbf{R}^3 onto \mathbf{R}^1 . This is true for almost all points of the \mathbf{R}^3 except the point $(0, 0, 0)$, since for this point the output of the function is not defined. Therefore the function is only defined on a domain D being subset of \mathbf{R}^3 , or written in a mathematical short term notation.

$$\phi(x, y, z) : D \subset \mathbf{R}^3 \mapsto \mathbf{R}^1$$

To become familiar with expression like equation 2.1 one can try to visualize them. We will discuss this in more detail in the next section.

2.1 Visualization of real-valued functions

Normally if we consider for example the simple function $f(x) = x^2$ we remember the picture shown in Figure 2.1. But is this picture identical with the function? Of course

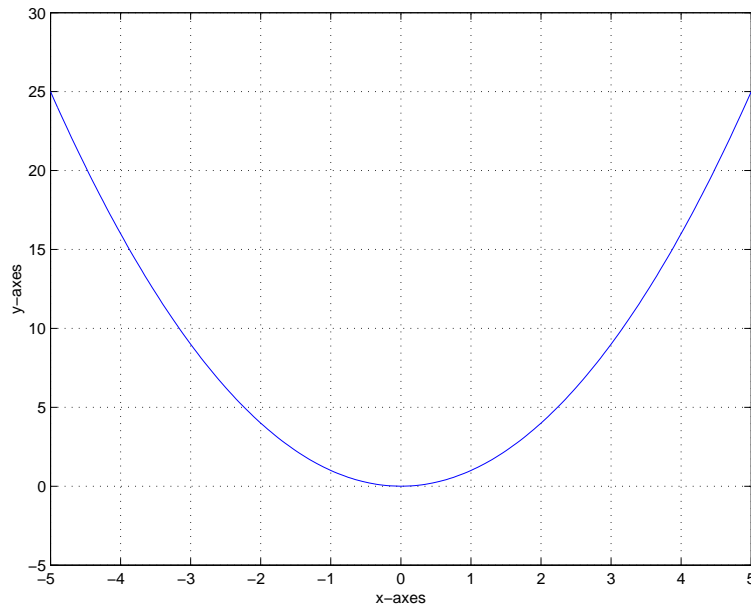


Figure 2.1: Graph of a simple function

not, since its only a representation of the function on a bounded domain or in other words Figure 2.1 tries only to visualize the function. The function itself is a mapping from \mathbf{R}^1 to \mathbf{R}^1 and from this operation it is hard to draw a picture. The picture shown in Figure 2.1 is a curve in \mathbf{R}^2 and to distinguish it from the function itself it is called the **graph** of the function. Whereas the function itself is a mapping from \mathbf{R}^1 to \mathbf{R}^1 the graph of the function is a mapping from \mathbf{R}^1 to \mathbf{R}^2 . In symbols, we write this as

$$\text{graph } f = \{(x, f(x)) \in \mathbf{R}^2 | x \in D \subset \mathbf{R}^1\}$$

where the curly braces mean *set of all* and the vertical bar is read *such that*. Drawing the graph of a function of one variable is a useful method to visualize how the function actually behaves. We will generalize the idea of a graph to real-valued functions of several variables.

Definition

Let $f : D \subset \mathbf{R}^n \mapsto \mathbf{R}$, we define the graph of f to be the subset of \mathbf{R}^{n+1} consisting of all points $(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) \in \mathbf{R}^{n+1}$ for $(x_1, x_2, \dots, x_n) \in D$.

For the case $n = 1$ the graph is a kurve in \mathbf{R}^2 , while for $n = 2$ it is a surface in \mathbf{R}^3 . For $n = 3$ the graph of a function is no longer visuable since it would be a 3-dimensional object in 4-dimensional space.

Example: As a first example Figure 2.2 shows the graph of the function $f(x, y) = x^2 + y^2$, which is a paraboloid in 3-dimensional space.

Example: As a second example we will have a view on the graph of the function $f(x, y) = x^2 - y^2$, which is shown in Figure 2.3 and looks like a saddle.

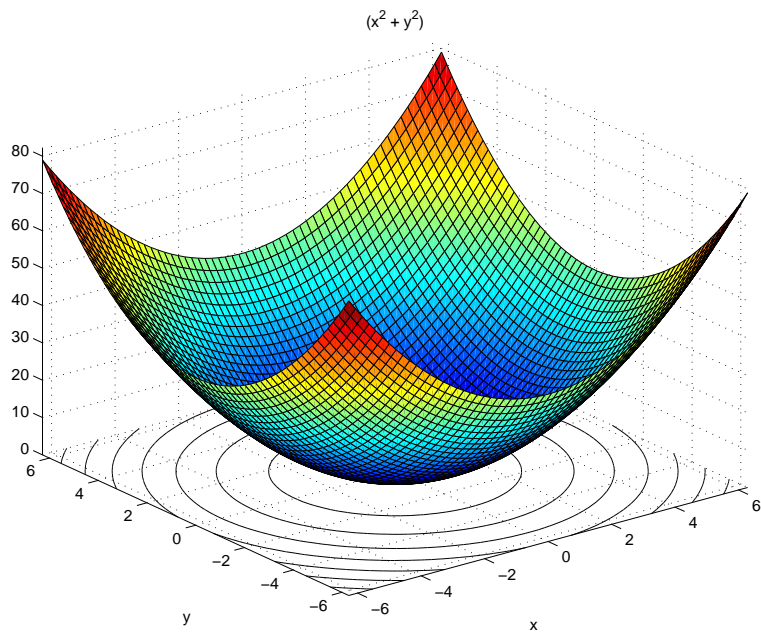


Figure 2.2: Graph of the function $f(x, y) = x^2 + y^2$

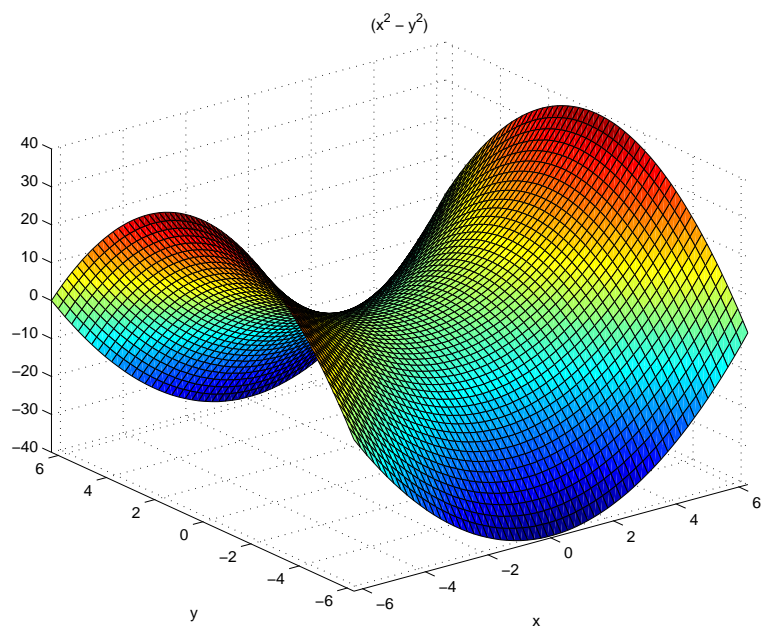


Figure 2.3: Graph of the function $f(x, y) = x^2 - y^2$

Another possibility to visualize a function is the so called level set. A level set of a function is defined as that subset in the source domain for which the function produces a constant output. If we again consider the function $f(x,y) = x^2 + y^2$ a level set in the source domain \mathbf{R}^2 is defined by $x^2 + y^2 = c$. As one knows these level curves are defined by all possible concentric circles in the plane. In Figure 2.2 the level curves are already shown in the xy -plane, which is the source domain of the function $f(x,y)$. Now after we have a crude understanding of a level set, we are able to give an exact definition.

Definition: Let $f(\vec{x}) : D \in \mathbf{R}^n \mapsto \mathbf{R}$ and let $c \in \mathbf{R}$. The level set of value c is defined to be those points $\vec{x} \in D$ at which $f(\vec{x}) = c$. If $n = 2$, we speak of a level curve and if $n = 3$ we speak of a level surface. In a mathematical notation the level set of value c is written as.

$$\{\vec{x} \in D \mid f(\vec{x}) = c\} \subset \mathbf{R}^n$$

2.2 Limits and continuity

In this section we will develop the terminology that will help us to study the differentiation of functions of several variables. We will introduce the concepts of the following terms:

- open disk,
- open sets,
- limits,
- continuity.

We will start our discussion with the definition of an open disk.

Definition: Let $\vec{x} \in \mathbf{R}^n$ and let r be a positive real number. The open disk (or open ball) of radius r and center at \vec{x}_0 is defined to be the set of all points \vec{x} such that $|\vec{x} - \vec{x}_0| < r$. This set is denoted $D_r(\vec{x}_0)$.

Notice that we include only those \vec{x} for which the *strict* inequality holds. The disk $D_r(\vec{x})$ is illustrated in Figure 2.4 for $n = 1, 2$. For the case $n = 1$ and $x_0 \in \mathbf{R}^1$, the open disk $D_r(x)$ is the open interval $(x_0 - r, x_0 + r)$. For the case $n = 2$, $\vec{x}_0 \in \mathbf{R}^2$, the open disk is the *inside* of the disk with radius r centered at \vec{x}_0 . For the case $n = 3$, $\vec{x}_0 \in \mathbf{R}^3$, $D_r(\vec{x})$ is the *inside* of the ball of radius r centered at \vec{x}_0 . With the help of the above definition we are ready to define an open set.

Definition: Let $U \in \mathbf{R}^n$ (that is let U be a subset of \mathbf{R}^n). We call U an **open set**, when for every point \vec{x}_0 there exists some $r > 0$ such that $D_r(\vec{x}_0)$ is contained within U .

Intuitively speaking, a set U is open, when the boundary points of U do not lie in U .

Theorem: For each $\vec{x}_0 \in \mathbf{R}^n$ and $r > 0$, $D_r(\vec{x}_0)$ is an open set.

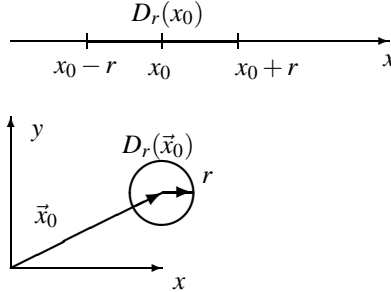


Figure 2.4: Open disk in 1,2-dimensional space

Proof: Let $\vec{x} \in D_r(\vec{x}_0)$, that is let $|\vec{x} - \vec{x}_0| < r$. According to the definition of an open set we must find a $s > 0$ such that $D_s(\vec{x}) \subset D_r(\vec{x}_0)$. Referring to Figure 2.4 we see that $s = r - |\vec{x} - \vec{x}_0|$ is a reasonable choice. Note that s is always greater than zero, but it becomes smaller if \vec{x} is nearer the edge of $D_r(\vec{x}_0)$. To prove that $D_s(\vec{x}) \subset D_r(\vec{x}_0)$ we consider a $\vec{y} \in D_s(\vec{x})$; that is $|\vec{y} - \vec{x}| < s$, hence we have:

$$|\vec{y} - \vec{x}_0| = |(\vec{y} - \vec{x}) + (\vec{x} - \vec{x}_0)| \leq |\vec{y} - \vec{x}| + |\vec{x} - \vec{x}_0| < s + |\vec{x} - \vec{x}_0| = r$$

Definition: A neighborhood of $\vec{x} \in \mathbf{R}^n$ is an open set U containing the point \vec{x} .

Definition: Let $A \subset \mathbf{R}^n$. A point $\vec{x} \in \mathbf{R}^n$ is called a boundary point of A if every neighborhood of \vec{x} contains at least one point in A and at least one point not in A .

With the above given definitions we are now ready to define the limit process in n-dimensional space.

Definition: Let $f : A \subset \mathbf{R}^n \mapsto \mathbf{R}^m$, where A is an open set. Let $\vec{x}_0 \in A$ or be a boundary point of A , and let N be a neighborhood of $\vec{b} \in \mathbf{R}^m$. We say f is eventually in N as \vec{x} approaches \vec{x}_0 if there exists a neighborhood U of \vec{x}_0 such that $\vec{x} \neq \vec{x}_0$, $\vec{x} \in U$, and $\vec{x} \in A$ imply $f(\vec{x}) \in N$.

We say $f(\vec{x})$ approaches \vec{b} as \vec{x} approaches \vec{x}_0 or in symbols.

$$\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = \vec{b}$$

To carry out practical computations with limits we require some additional rules which are given without proof.

Uniqueness of limits

$$\text{If } \lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b}_1 \quad \text{and} \quad \lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b}_2 \quad \text{then} \quad \vec{b}_1 = \vec{b}_2$$

Let $\vec{f} : A \subset \mathbf{R}^n \mapsto \mathbf{R}^m$, $\vec{g} : A \subset \mathbf{R}^n \mapsto \mathbf{R}^m$ and \vec{x}_0 be in A or a boundary point of A and

$\vec{b} \in \mathbf{R}^m$ and $c \in \mathbf{R}^1$ then the following theorems hold true:

If $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b}$ then $\lim_{\vec{x} \rightarrow \vec{x}_0} c\vec{f}(\vec{x}) = c\vec{b}$

If $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b}_1$ and $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{g}(\vec{x}) = \vec{b}_2$ then $\lim_{\vec{x} \rightarrow \vec{x}_0} (\vec{f}(\vec{x}) + \vec{g}(\vec{x})) = \vec{b}_1 + \vec{b}_2$

If $m = 1$ $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b}_1$ and $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{g}(\vec{x}) = c\vec{b}_2$
then $\lim_{\vec{x} \rightarrow \vec{x}_0} (\vec{f}(\vec{x}) \cdot \vec{g}(\vec{x})) = \vec{b}_1 \cdot \vec{b}_2$

If $m = 1$ $\lim_{\vec{x} \rightarrow \vec{x}_0} f(\vec{x}) = b \neq 0$ and $f(\vec{x}) \neq 0$ for all $\vec{x} \in A$ then $\lim_{\vec{x} \rightarrow \vec{x}_0} 1/f = 1/b$

If $\vec{f}(\vec{x}) = (f_1(\vec{x}), \dots, f_m(\vec{x}))$ where $f_i : A \mapsto \mathbf{R}^1, i = 1, \dots, m$,
are the component functions of f , then $\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{b} = (b_1, \dots, b_m)$ if and only if
 $\lim_{\vec{x} \rightarrow \vec{x}_0} f_i(\vec{x}) = b_i$ for each $i = 1, \dots, m$

(2.2)

With the help of the limit process, we are now able to define the **continuity** of a function.

Definition: Let $\vec{f} : A \subset \mathbf{R}^n \mapsto \mathbf{R}^m$ be a given vector valued function with domain A . Let $\vec{x}_0 \in A$. We say \vec{f} is continuous at \vec{x}_0 if and only if

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \vec{f}(\vec{x}) = \vec{f}(\vec{x}_0)$$

If we just say that \vec{f} is continuous, we shall mean that \vec{f} is continuous at each point \vec{x}_0 of A . Since the continuity of a function is defined by the limit process there exist equations equivalent to 2.2 for the continuity of functions.

2.3 Differentiation of functions

Intuitively, we know from our work in the last section that a continuous function is one that has no breaks in its graph. A differentiable function from \mathbf{R}^2 to \mathbf{R}^1 ought to be such that not only there are no breaks in its graph, but there is a well defined plane tangent to the graph at each point. Thus, there must not be any sharp fold, corners, or peaks in the graph. In other words the graph must be smooth.

Before we can define the differentiability of a real valued function $f(x_1, \dots, x_n)$ at the point $\vec{x} = (x_1, \dots, x_n)$ we will introduce the notion of the *partial derivate*. To do this we will first have a look at the term of differentiability in one dimensional calculus. Figure 2.5 shows the graph of a one dimensional real valued function. The term differentiability of the function is connected with the geometric problem to construct a tangent to the graph at a given point x as shown in the Figure. This is done by calculating the value of the function at two slightly different points $x+h$ and x , and dividing the difference of the function values by the difference of the variables. Introducing the limit process $\lim_{h \rightarrow 0}$ yields the known definition for the differentiation of a one dimensional function.

$$\frac{df(x)}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

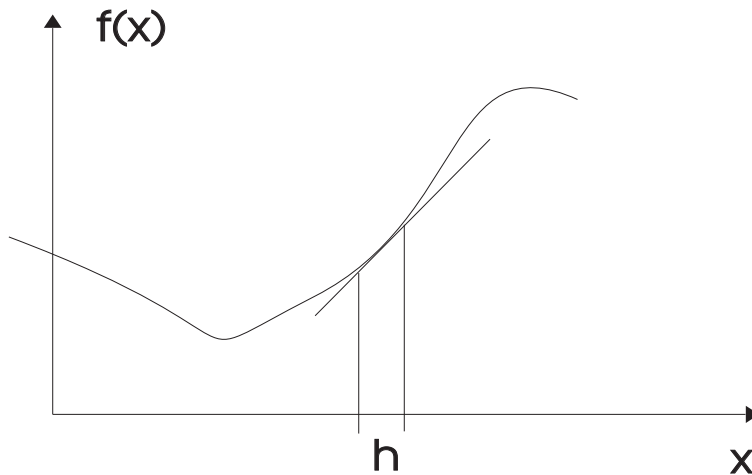


Figure 2.5: Solving the tangent problem

With the help of this repetition one can do the definition of the partial derivative.

Definition Let $f : U \subset \mathbf{R}^n \mapsto \mathbf{R}^1$ be a real valued function of n variables. Then the partial derivatives $\partial f / \partial x_i$ with respect to the i -th variable x_i is defined by

$$\frac{\partial f}{\partial x_i} = \lim_{h \rightarrow 0} \frac{f(x_1, \dots, x_i + h, \dots, x_n) - f(x_1, \dots, x_i, \dots, x_n)}{h} \quad (2.3)$$

From the above definition it is clear, that the calculation of the partial derivative is essentially done by doing a "normal" derivation with respect to one variable treating the remaining variables as constants.

Example: Given the function $f(x, y) = x^2y + y^3$. Find $\partial f / \partial x$ and $\partial f / \partial y$

Solution

$$\frac{\partial f}{\partial x} = 2xy \quad \frac{\partial f}{\partial y} = x^2 + 3y^2$$

To indicate that a partial derivate is to be evaluated at a particular point, for example (x_0, y_0) we write

$$\frac{\partial f}{\partial x}(x_0, y_0) \quad \text{or} \quad \left. \frac{\partial f}{\partial x} \right|_{x=x_0, y=y_0} \quad \text{or} \quad \left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)}$$

Example: $f(x, y) = \cos(xy) + x \cos(y)$. Find the partial derivatives $\partial f / \partial x(x_0, y_0)$ and $\partial f / \partial y(x_0, y_0)$.

First we fix $y = y_0$ and differentiate with respect to x , giving

$$\frac{\partial f}{\partial x}(x_0, y_0) = \left. \frac{d[\cos(xy_0) + x \cos(y_0)]}{dx} \right|_{x=x_0} = -y_0 \sin(x_0 y_0) + \cos(y_0)$$

Similary we next fix $x = x_0$ and differentiate with respect to y to obtain

$$\frac{\partial f}{\partial y}(x_0, y_0) = \left. \frac{d[\cos(x_0 y) + x_0 \cos(y)]}{dy} \right|_{x=x_0} = -x_0 \sin(x_0 y_0) - x_0 \sin(y_0)$$

Strange Example: Given the function $f(x,y) = x^{1/3}y^{1/3}$. Find the partial derivate $\partial f/\partial x$ at $(0,0)$.

If we just do the differentiation with respect to x we find

$$\frac{\partial f}{\partial x} = \frac{1}{3}x^{-2/3}y^{1/3}$$

for the given point $(0,0)$ this function is not defined, in contrast to the result when we start from the definition 2.3

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

Here we have the case that the functions $x^{1/3}$ and $y^{1/3}$ are themselves not differentiable at 0 but the multiplication of the functions has partial derivatives in this point. To overcome this problem we have to construct something that is in n -dimensional space equivalent to the tangent line in one dimensional space.

First we will focus our discussion on real valued functions $f : U \subset \mathbf{R}^2 \mapsto \mathbf{R}^1$. In \mathbf{R}^3 a nonvertical plane for example is described by the following equation

$$z = ax + by + c$$

If it is to be the plane tangent to the graph of a function $f(x,y)$ the slopes along the x and y -axes must be equal to $\partial f/\partial x$ and $\partial f/\partial y$. Thus we have $a = \partial f/\partial x$ and $b = \partial f/\partial y$ both evaluated at a given point (x_0, y_0) and the constant c will be given by $f(x_0, y_0)$, so we have

$$z = f(x_0, y_0) + \left. \frac{\partial f}{\partial x} \right|_{x_0, y_0} (x - x_0) + \left. \frac{\partial f}{\partial y} \right|_{x_0, y_0} (y - y_0) \quad (2.4)$$

This is the equation of the plane tangent to the graph of the function $f(x,y)$ at a given point (x_0, y_0) if $f(x,y)$ is "smooth enough". The definition of differentiability will mean in fact, that the plane given by the above equation is a "good" approximation of the function $f(x,y)$ near a given point (x_0, y_0) . In the one dimensional case the differentiability was written as

$$\frac{df}{dx} = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

one can rewrite this like

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} - \frac{df}{dx} = 0 \quad \text{or} \quad \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0) - \frac{df}{dx}(x - x_0)}{x - x_0} = 0$$

With the help of the above equations we are ready to define the differentiability for functions $\mathbf{R}^2 \mapsto \mathbf{R}^1$

Definition Let $f : \mathbf{R}^2 \mapsto \mathbf{R}^1$, we say f is differentiable at (x_0, y_0) if $\partial f/\partial x$ and $\partial f/\partial y$ exist at (x_0, y_0) and if

$$\lim_{x \rightarrow x_0, y \rightarrow y_0} \frac{f(x,y) - f(x_0, y_0) - \frac{\partial f}{\partial x}(x - x_0) - \frac{\partial f}{\partial y}(y - y_0)}{|(x,y) - (x_0, y_0)|} \rightarrow 0$$

This equation expresses what we mean by saying that

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0)$$

is a "good approximation" to the function $f(x, y)$. So, if the function $f(x, y)$ is differentiable at (x_0, y_0) one can construct the tangent plane at this point with the help of equation 2.4.

Example: Compute the plane tangent to the graph of the function $f(x, y) = x^2 + y^2 + e^{xy}$ at the point $(1, 0)$.

The partial derivatives and $f(x_0, y_0)$ at $(1, 0)$ are

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x + ye^{xy}|_{(1,0)} = 2 \\ \frac{\partial f}{\partial y} &= 2y + xe^{xy}|_{(1,0)} = 1 \\ f(x_0, y_0) &= 1 + 0 + 1 = 2 \end{aligned}$$

So we find as equation for the tangent plane

$$z = 2 + 2(x - 1) + 1(y - 0)$$

Now we are ready to extend the definition of differentiability to real valued functions defined on spaces with arbitrary dimension n .

Definition : Let $f : U \subset \mathbf{R}^n \mapsto \mathbf{R}^1$, we say f is differentiable at \vec{x}_0 if all partial derivatives $\partial f / \partial x_1, \partial f / \partial x_2, \dots, \partial f / \partial x_n$ exist and if

$$\lim_{\vec{x} \rightarrow \vec{x}_0} \frac{f(\vec{x}) - f(\vec{x}_0) - \vec{\nabla} f \cdot (\vec{x} - \vec{x}_0)}{|\vec{x} - \vec{x}_0|} \rightarrow 0 \quad \text{with} \quad \vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \quad \text{called gradient of } f$$

A function whose partial derivatives exist and are continuous is said to be of class \mathbf{C}^1 . Any \mathbf{C}^1 function is differentiable. In the 2-dimensionale case the gradient is normally written as

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \vec{u}_x \frac{\partial f}{\partial x} + \vec{u}_y \frac{\partial f}{\partial y}$$

with \vec{u}_x and \vec{u}_y being unit vectors in the x-, respectively in the y-direction. In the case of a 3-dimensional cartesian coordinate system the gradient of a function looks like

$$\vec{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \vec{u}_x \frac{\partial f}{\partial x} + \vec{u}_y \frac{\partial f}{\partial y} + \vec{u}_z \frac{\partial f}{\partial z} \quad (2.5)$$

Normally the symbolic vector

$$\vec{\nabla} = \vec{u}_x \frac{\partial}{\partial x} + \vec{u}_y \frac{\partial}{\partial y} + \vec{u}_z \frac{\partial}{\partial z}$$

is called **nabla-operator**. The form given above is its representation in a cartesian coordinate system.

2.4 Gradients and directional derivate

In forthcoming section we will discuss some special properties of the gradient operation. As one can see from equation 2.5 the gradient operation is equivalent to a mapping from \mathbf{R}^1 to \mathbf{R}^3 . We will start the discussion of the properties of the resulting vector by considering the following scalar function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2} = |\vec{r}|$$

This function describes the distance from the origin in a cartesian coordinate system, what is identical with the absolute value of the radius vector in a spherical coordinate system. To calculate the gradient of the above defined function we start with the x-component of the vector

$$\frac{\partial f}{\partial x} = \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2} 2x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

hence we get for the total gradient

$$\vec{\nabla} f = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{\vec{r}}{|\vec{r}|} = \vec{u}_r$$

As we see from the above equation the gradient of the distance function is equal to \vec{u}_r , the unit vector in radius direction of the spherical coordinate system. It is interesting to notice that the gradient of the function f is normal to its level surfaces $f(x, y, z) = c$. Later on we will proof that this is a general property of the gradient.

Example: The static electric field \vec{E} is given by the negativ gradient of the potential function. Calculate the electric field using the potential function 2.1.

It is sufficient to first evaluate the partial derivative with respect to the x-coordinate

$$\frac{\partial \phi}{\partial x} \sim \frac{\partial(x^2 + y^2 + z^2)^{-1/2}}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2} 2x$$

Hence we get for the total electric field

$$\vec{E} = -\vec{\nabla} \phi = -\frac{Q}{4\pi\epsilon_0} \vec{\nabla} \left(\frac{1}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{Q}{4\pi\epsilon_0} \frac{1}{(\sqrt{x^2 + y^2 + z^2})^3} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or with the help of the radius vector of the spherical coordinate system

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{\vec{r}}{|\vec{r}|^3} = \frac{Q}{4\pi\epsilon_0} \frac{1}{|\vec{r}|^2} \vec{u}_r$$

Now we want to study a real valued function $f: U \subset \mathbf{R}^3 \mapsto \mathbf{R}^1$ along a straight line with direction \vec{u}_v . A straight line may be defined by a constant vector \vec{r}_0 and a parameter t .

$$\vec{r}(t) = \vec{r}_0 + \vec{u}_v t$$

Using this vector we can calculate the function $f(\vec{r})$ along the straight line $\vec{r}(t)$. In this case the function f will only be dependent on the parameter t and the directional derivative in the direction of the unit vector \vec{u}_v will be given by

$$\left. \frac{df(\vec{r}_0 + \vec{u}_v t)}{dt} \right|_{t=0}$$

In a cartesian coordinate system we have

$$\vec{r}(t) = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \begin{pmatrix} u_{v_x} \\ u_{v_y} \\ u_{v_z} \end{pmatrix} t$$

normally the function f is given as $f(x, y, z)$. So if we want to calculate the derivative with respect to the parameter t , we have to use the chain rule

$$\begin{aligned} \frac{df(x(t), y(t), z(t))}{dt} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t} = \\ &= \frac{\partial f}{\partial x} u_{v_x} + \frac{\partial f}{\partial y} u_{v_y} + \frac{\partial f}{\partial z} u_{v_z} = (\vec{\nabla} f) \cdot \vec{u}_v \end{aligned}$$

Therefore we find that the derivative of a function f in the direction of a unit vector \vec{u}_v is given by

$$\left. \frac{df(\vec{r}_0 + \vec{u}_v t)}{dt} \right|_{t=0} = (\vec{\nabla} f) \cdot \vec{u}_v$$

Example: Given $f(x) = x^2 \exp(-yz)$, compute the rate of change in the direction of the unit vector $\vec{u}_v = (1, 1, 1)/\sqrt{3}$ at $\vec{r}_0 = (1, 0, 0)$

For the gradient of the function we find

$$\vec{\nabla}[x^2 \exp(-yz)] = 2x \exp(-yz) \vec{u}_x - x^2 z \exp(-yz) \vec{u}_y - x^2 y \exp(-yz) \vec{u}_z \Big|_{(1,0,0)} = 2\vec{u}_x$$

hence we get

$$(\vec{\nabla} f) \cdot \vec{u}_v = \frac{2}{\sqrt{3}}$$

We will now show that the gradient of a function points in the direction along which f is increasing the fastest. To do this we assume an arbitrary unit vector \vec{u} . Then the rate of change of f in direction \vec{u} is given by

$$(\vec{\nabla} f) \cdot \vec{u} = |\vec{\nabla} f| \cos(\theta)$$

where θ is the angle between \vec{u} and $\vec{\nabla} f$. This becomes a maximum when $\theta = 0$, hence the fastest increase of f is in the direction of $\vec{\nabla} f$. In other words if one wishes to move in a direction in which f will increase most quickly one should proceed in the direction of the gradient of f . Similar if one wishes to move in a direction in which f decreases fastest, one should proceed in the direction of $-\vec{\nabla} f$.

If we consider a arbitrary scalar function $f : U \subset \mathbf{R}^3 \mapsto \mathbf{R}^1$ and if we consider a point $\vec{r}_0 = (x_0, y_0, z_0)$ lying on a level surface defined by $f(x, y, z) = c$, then the gradient of f at $\vec{r}_0 = (x_0, y_0, z_0)$ is normal to the level surface. This is clear if we think of a unit vector \vec{u} lying in the level surface. The rate of change in a level surface is zero, hence we have

$$(\vec{\nabla} f) \cdot \vec{u} = 0 \quad \Rightarrow \quad \vec{\nabla} f \perp \vec{u}$$

Chapter 3

Vector-valued functions

One of our main concerns in chapter 2 was the study of real-valued functions. This chapter deals with functions whose values are vectors. We will start our discussion with paths, which are maps from \mathbf{R}^1 onto \mathbf{R}^n .

3.1 Paths

One often thinks of a curve as a line drawn on a piece of paper, such as a straight line, a circle, or a sine curve. To deal with such objects effectively it is convenient to think of a curve in \mathbf{R}^n as a set of values of a function which maps an interval from \mathbf{R}^1 onto \mathbf{R}^n . If we formulate this for example for a path in 3-dimensional space it looks like

$$t \in [a, b] \mapsto \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

If we think of the parameter t being the time, then $\vec{r}(t)$ could be thought of being a vector pointing to a place in space where for example a mass point is moving along the path.

Definition: A path in \mathbf{R}^n is a map $\vec{r}(t) : t \in [a, b] \mapsto \mathbf{R}^n$. If $\vec{r}(t)$ is differentiable, we say the mapping is a differentiable path. If \vec{r} is of class \mathbf{C}^1 , we say \vec{r} is a \mathbf{C}^1 path. The points $\vec{r}(a)$ and $\vec{r}(b)$ are called the endpoints of the path.

Example: A straight line $\vec{r}(t)$ in \mathbf{R}^3 thru a point $\vec{r}_0 = (x_0, y_0, z_0)$ in the direction of a vector \vec{v} is given by:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}t$$

Example: A circle in the xy -plane with radius ρ_0 is represented by

$$\vec{r}(\varphi) = \rho_0 \begin{pmatrix} \sin(\varphi) \\ \cos(\varphi) \end{pmatrix} \quad \varphi \in [0, 2\pi] \quad (3.1)$$

Example: A helix in 3-dimensional space is given by

$$\vec{r} = \rho_0 \cos(\omega t) \vec{u}_x + \rho_0 \sin(\omega t) \vec{u}_y + v_0 t \vec{u}_z \quad (3.2)$$

Usually particles that move in space do so on smooth curves. For example particles do

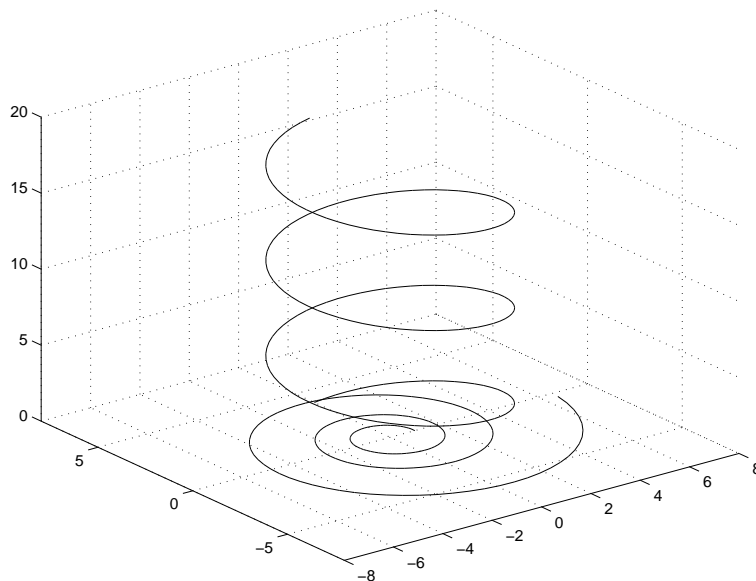


Figure 3.1: Path of a helix and a spiral

not usually disappear and spontaneously reappear at another point or change their velocity suddenly. Thus we shall restrict our attention to sufficiently smooth paths, say \mathbf{C}^1 for the rest of this section.

Definition: Let $\vec{r}(t) : \mathbf{R}^1 \mapsto \mathbf{R}^3$ be a \mathbf{C}^1 path. The velocity vector $\vec{v}(t)$ at a certain point $\vec{r}(t)$ is then given by

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \begin{pmatrix} \frac{dx(t)}{dt} \\ \frac{dy(t)}{dt} \\ \frac{dz(t)}{dt} \end{pmatrix}$$

and the total speed of the particle is $v = |\vec{v}|$.

As an example we calculate the velocity of a particle moving along the helix given by equation 3.2

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \begin{pmatrix} -\rho_0\omega \sin(\omega t) \\ \rho_0\omega \cos(\omega t) \\ v_0 \end{pmatrix}$$

and for the total speed $v = |\vec{v}|$, we get

$$v = \sqrt{(\rho_0\omega)^2 \cos^2(\omega t) + (\rho_0\omega)^2 \sin^2(\omega t) + v_0^2} = \sqrt{(\rho_0\omega)^2 + v_0^2} \quad (3.3)$$

The following equations give differentiation rules for paths $\vec{r}(t) : \mathbf{R}^1 \mapsto \mathbf{R}^n$

$$\begin{aligned} \frac{d}{dt}(\vec{a} + \vec{b}) &= \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt} \\ \frac{d}{dt}(\lambda\vec{a}) &= \frac{d\lambda}{dt}\vec{a} + \lambda\frac{d\vec{a}}{dt} \end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(\vec{a} \cdot \vec{b}) &= \frac{d\vec{a}}{dt} \cdot \vec{b} + \vec{a} \cdot \frac{d\vec{b}}{dt} \\ \frac{d}{dt}(\vec{a} \times \vec{b}) &= \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt} \quad \vec{a}, \vec{b} \in \mathbf{R}^3\end{aligned}$$

3.2 Arc length

Consider a given path $\vec{r}(t)$, we can think of it as the path of a particle. The particle travels along that curve in space. What is the length of this curve as t ranges from a to b . To calculate the length s of the curve we think of the curve being subdivided into infinitesimal small sections $ds = |d\vec{r}|$ and sum them up along the total curve. This yields the following equation

$$s = \int ds = \int_a^b \left| \frac{d\vec{r}}{dt} \right| dt \quad (3.4)$$

In the case of paths in \mathbf{R}^2 or in \mathbf{R}^3 the above equation specializes to

$$\begin{aligned}s &= \int_a^b \left| \frac{d\vec{r}}{dt} \right| dt = \int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2} dt \\ s &= \int_a^b \left| \frac{d\vec{r}}{dt} \right| dt = \int_a^b \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2} dt\end{aligned}$$

Example: Calculate the arc length of the curve given by equation 3.1 for $\varphi \in [0, \pi]$. The vector $d\vec{r}/d\varphi$ is given by

$$\frac{d\vec{r}}{d\varphi} = \rho_0 \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$

hence we get for its absolute value

$$\left| \frac{d\vec{r}}{d\varphi} \right| = \sqrt{\rho_0^2 [\sin^2(\varphi) + \cos^2(\varphi)]} = \rho_0$$

and for the arc length

$$s = \int_0^\pi \left| \frac{d\vec{r}}{d\varphi} \right| d\varphi = \int_0^\pi \rho_0 d\varphi = \pi \rho_0$$

Example: Find the arc length of the helix defined by equation 3.2 and $t \in [0, T]$.

We have already calculated the speed of a particle along the helix resulting in equation 3.3, therefore we find for the arc length

$$s = \int_0^T \left| \frac{d\vec{r}}{dt} \right| dt = \int_0^T \sqrt{(\rho_0 \omega)^2 + v_0^2} dt = \sqrt{(\rho_0 \omega)^2 + v_0^2} T$$

3.2.1 Natural parametrization

So far we consider any arbitrary parametrization of a path $\vec{r}(t)$ with $t \in [a, b]$. With the help of equation 3.4 it is possible to define an arc-length function $s(t)$ on the path.

$$s(t) = \int_a^t \left| \frac{d\vec{r}(\tau)}{d\tau} \right| d\tau \quad t \in [a, b]$$

So $s(t)$ defines a mapping from the interval $[a, b] \in \mathbf{R}^1$ of the parameter t to an interval $[0, L] \in \mathbf{R}^1$ of the parameter s . If there exists the inverse mapping $t(s)$ we can define the natural parametrisation of a path, for example in \mathbf{R}^3

$$\vec{r}(s) = \begin{pmatrix} x(s) \\ y(s) \\ z(s) \end{pmatrix}$$

To clarify the above consideration we examine the following path, which describes a partial moving in the xy-plane on a circle with fixed radius ρ .

$$\vec{r}(t) = \rho_0 \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix} \quad \text{for } t \in [0, T]$$

To evaluate the function $s(t)$, we first have to evaluate $|d\vec{r}/dt|$

$$\frac{d\vec{r}(t)}{dt} = \rho_0 \omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix} \rightarrow \left| \frac{d\vec{r}(t)}{dt} \right| = \rho_0 \omega$$

so we find for the arc length function $s(t)$

$$s(t) = \int_0^t \rho_0 \omega dt = \rho_0 \omega t$$

In this case it is easy to find the inverse function $t = s/(\rho_0 \omega)$. We are now able to give a natural parametrization of the path.

$$\vec{r}(s) = \rho_0 \begin{pmatrix} \cos(s/\rho_0) \\ \sin(s/\rho_0) \end{pmatrix} \quad \text{for } s \in [0, L]$$

Definition: A \mathbf{C}^1 parametrization of a path $\vec{r}(s) : \mathbf{R}^1 \mapsto \mathbf{R}^n$ is a natural or path-length parametrization if we have

$$\left| \frac{d\vec{r}}{ds} \right| = 1 \quad \text{or} \quad |d\vec{r}| = ds$$

We give the following theorem without proof.

Theorem: For any smooth parametrization $\vec{r} : \mathbf{R}^1 \mapsto \mathbf{R}^n$ of a path in \mathbf{R}^n there exists a natural parametrization.

3.3 Differential geometry of a path

In this section we will consider paths with a natural parametrization in a 3-dimensional space.

$$s \in [0, L] \mapsto \vec{r}(s) \in \mathbf{R}^3$$

First we will calculate the tangent vector to the path

$$\vec{T}(s) = \frac{d\vec{r}}{ds}$$

for the length of the tangent vector we find

$$|\vec{T}(s)| = \left| \frac{d\vec{r}}{ds} \right| = \frac{ds}{ds} = 1$$

This shows that the tangent vector of a natural parametrized path is a unit vector and we have $\vec{T}(s) \cdot \vec{T}(s) = 1$. If we differentiate this expression with respect to the parameter s we find

$$\frac{d}{ds}[\vec{T}(s) \cdot \vec{T}(s)] = 0 \Rightarrow \frac{d\vec{T}(s)}{ds} \cdot \vec{T}(s) = 0$$

If these two vectors do exist they are orthogonal to each other.

Definition: The non negative number

$$\kappa(s) = \left| \frac{d\vec{T}(s)}{ds} \right|$$

is called the curvature of $\vec{r}(s)$ in s .

Definition: If $\kappa(s) \neq 0$ then a unit vector $\vec{N}(s)$ such that

$$\frac{d\vec{T}(s)}{ds} = \kappa(s)\vec{N}(s)$$

is called the principal normal to $\vec{r}(s)$.

Definition: The vector $\vec{B}(s) = \vec{T}(s) \times \vec{N}(s)$ is called the binormal unit vector.

If we differentiate \vec{B} with respect to s we find

$$\frac{d\vec{B}}{ds} = \frac{d}{ds}(\vec{T} \times \vec{N}) = \vec{T} \times \frac{d\vec{N}}{ds}$$

Since \vec{B} is a unit vector $\frac{d\vec{B}}{ds}$ is normal to the vector \vec{B} , that is $\frac{d\vec{B}}{ds}$ is lying in the plane spanned by the vectors \vec{N} and \vec{T} . Since the vector $\vec{T} \times \frac{d\vec{N}}{ds}$ is normal to \vec{T} we have

$$\frac{d\vec{B}}{ds} = -\tau(s)\vec{N}(s)$$

The number $\tau(s)$ is called the torsion of $\vec{r}(s)$.

3.4 Arbitrary orthogonal coordinates

3.4.1 Cylindrical coordinates

The standard way to represent a point in a plane \mathbf{R}^2 is with the help of cartesian coordinates. However cartesian coordinates are not always very suitable for example if one wants to describe a circle polar coordinates are more appropriate. As shown in Figure 3.2 an arbitrary point (x, y) in the plane may also be represented by the radius ρ and the angle φ of the point (x, y) . So we have the following relations between the coordinates

$$x = \rho \cos(\varphi) \quad y = \rho \sin(\varphi)$$

We can use the polar coordinates as basis to introduce cylindrical coordinates and to represent a arbitrary point (x, y, z) in \mathbf{R}^3 .

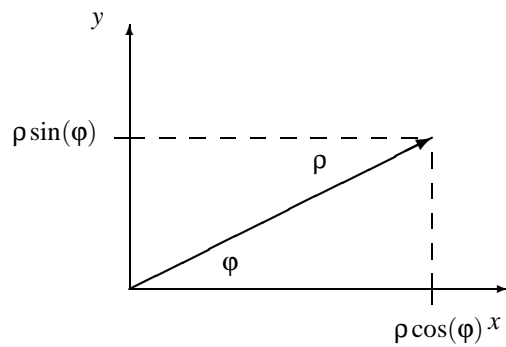


Figure 3.2: Definition of polar coordinates

Definition: The cartesian coordinates of a point given by its cylindrical coordinate values (ρ, φ, z) may be calculated by:

$$x = \rho \cos(\varphi) \quad y = \rho \sin(\varphi) \quad z = z \quad (3.5)$$

The values of the variables (ρ, φ, z) of the cylindrical coordinate system are restricted

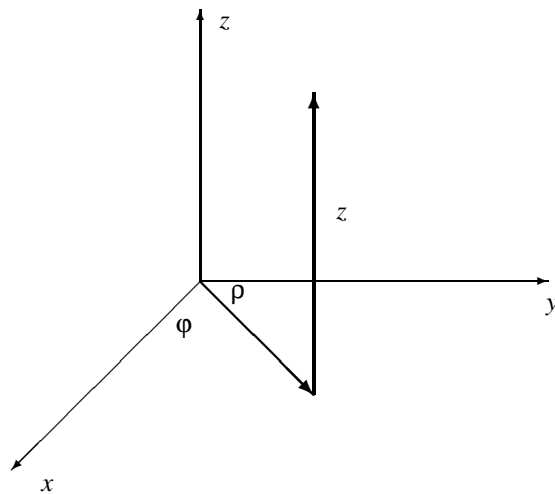


Figure 3.3: Definition of cylindrical coordinates

to the following domains

$$0 \leq \rho < \infty \quad 0 \leq \varphi < 2\pi \quad -\infty < z < \infty$$

If only one of the variables is changed one gets the so called coordinate lines of this variable. Figure 3.4 shows a point ρ_0, φ_0, z_0 in 3-dimensional coordinate system and around this point the ρ -, φ - and z -coordinate lines are drawn which are given by the

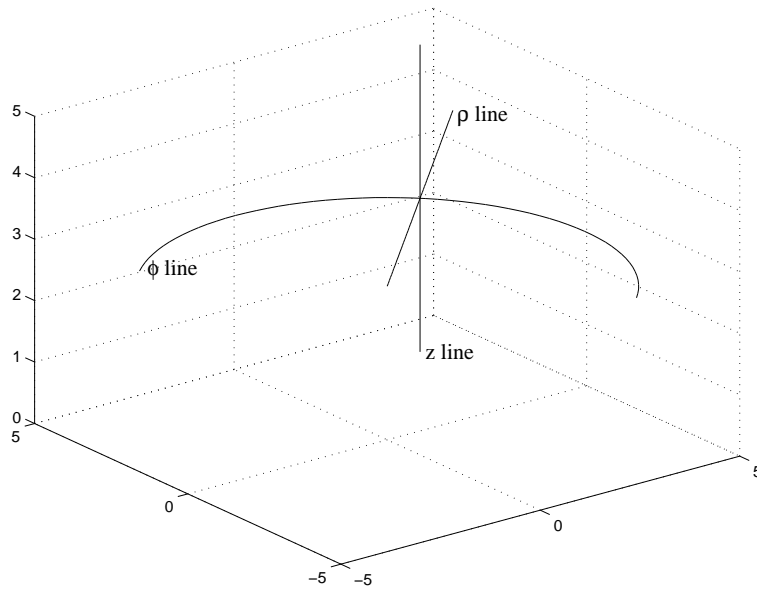


Figure 3.4: Coordinate lines of the cylindric coordinate system

following equations

$$\begin{aligned}
 \vec{r}(\Delta\rho) &= \begin{pmatrix} (\rho_0 + \Delta\rho) \cos(\varphi) \\ (\rho_0 + \Delta\rho) \sin(\varphi) \\ z_0 \end{pmatrix} && \rho\text{-coordinate line} \\
 \vec{r}(\Delta\varphi) &= \begin{pmatrix} \rho_0 \cos(\varphi + \Delta\varphi) \\ \rho_0 \sin(\varphi + \Delta\varphi) \\ z_0 \end{pmatrix} && \varphi\text{-coordinate line} \\
 \vec{r}(\Delta z) &= \begin{pmatrix} \rho_0 \cos(\varphi) \\ \rho_0 \sin(\varphi) \\ z_0 + \Delta z \end{pmatrix} && z\text{-coordinate line}
 \end{aligned} \tag{3.6}$$

In the subsequent section we will discuss the unit vectors of the cylindrical coordinate system. We will start our discussion with the z -coordinate line 3.6. The tangent vector to this line is given by

$$\frac{d\vec{r}(z)}{dz} = \frac{d}{dz} \begin{pmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The unit vector has the same direction in each point of the 3-dimensional space. Now we will calculate the unit vector in the ρ -direction. The tangent vector to the ρ -coordinate lines is given by

$$\frac{d\vec{r}(\rho)}{d\rho} = \frac{d}{d\rho} \begin{pmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ z \end{pmatrix} = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix}$$

Since we have $\cos^2(\varphi) + \sin^2(\varphi) = 1$ the tangent vector has already a unit length, hence we already found the unit vector \vec{u}_ρ of the cylinder coordinate system

$$\vec{u}_\rho = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} = \vec{u}_x \cos(\varphi) + \vec{u}_y \sin(\varphi) \quad (3.7)$$

In contrast to the unit vector \vec{u}_z the unit vector \vec{u}_ρ changes its direction depending on the point one considers in the xy -plane as illustrated in Figure 3.5. In a last step we will now construct the unit vector in the φ -direction. Its tangent vector is calculated by

$$\frac{\vec{r}(\varphi)}{d\varphi} = \frac{d}{d\varphi} \begin{pmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ z \end{pmatrix} = \begin{pmatrix} -\rho \sin(\varphi) \\ \rho \cos(\varphi) \\ 0 \end{pmatrix}$$

In contrast to the already mentioned tangent vector its norm is not constant

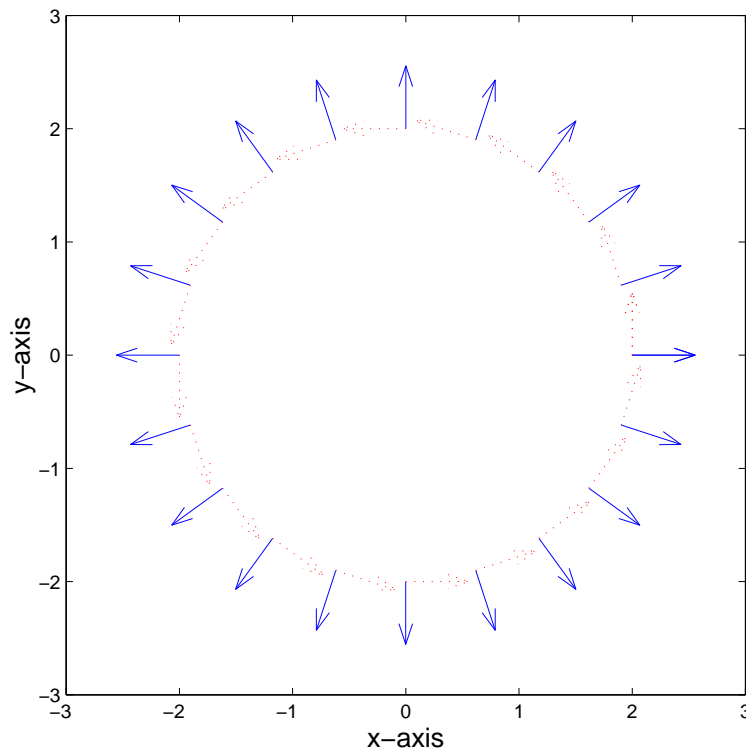


Figure 3.5: Unit vectors of the cylinder coordinate system

$$\left| \frac{\vec{r}(\varphi)}{d\varphi} \right| = \sqrt{\rho^2 \sin^2(\varphi) + \rho^2 \cos^2(\varphi)} = \rho$$

hence the unit vector \vec{u}_φ is given by

$$\vec{u}_\varphi = \frac{\frac{d\vec{r}}{d\varphi}}{\left| \frac{d\vec{r}}{d\varphi} \right|} = \frac{1}{\rho} \begin{pmatrix} -\rho \sin(\varphi) \\ \rho \cos(\varphi) \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} = -\vec{u}_x \sin(\varphi) + \vec{u}_y \cos(\varphi) \quad (3.8)$$

This vector is also shown in Figure 3.5 at different points of the $\rho\phi$ -plane. If we compare the unit vectors \vec{u}_ρ and \vec{u}_ϕ with the unit vectors \vec{u}_x and \vec{u}_y of the cartesian coordinate system we find that the unit vectors of the cylindrical coordinate system change their direction depending on the point we consider as shown in Figure 3.5.

3.4.2 Spherical coordinates

Beside cylindrical coordinates, spherical coordinates are often used in technical applications. Especially in the case of spherical symmetry e.g. the radia from a point source they can be applied with benefit. In Figure 3.6 the three quantities are defined that are used to describe a point in 3-dimensional space. These are the distant $|\vec{r}|$ of the

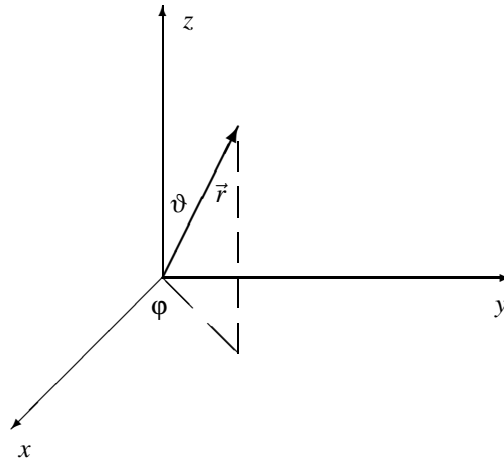


Figure 3.6: Definition of spherical coordinates

point from the origin, the angle ϑ between the z-axes and the vector \vec{r} and the angle φ between the x-axes and the projection of \vec{r} onto the xy-plane. With the help of Figure 3.6 one can easily define the mapping from spherical to cartesian coordinates.

Definition: The cartesian coordinates of a point given by its spherical values (r, ϑ, φ) may be calculated by:

$$x = r \sin(\vartheta) \cos(\varphi) \quad y = r \sin(\vartheta) \sin(\varphi) \quad z = r \cos(\vartheta)$$

where the parameter (r, ϑ, φ) are out of the following domains

$$0 \leq r < \infty \quad 0 \leq \vartheta \leq \pi \quad 0 \leq \varphi < 2\pi$$

To study the unit vectors of this coordinate system we have to calculate the tangent vectors to the different coordinate lines. The vector to an arbitrary point given by its spherical coordinate values (r, ϑ, φ) reads

$$\vec{r}(r, \vartheta, \varphi) = \begin{pmatrix} r \sin(\vartheta) \cos(\varphi) \\ r \sin(\vartheta) \sin(\varphi) \\ r \cos(\vartheta) \end{pmatrix} \quad (3.9)$$

while Figure 3.7 shows coordinate lines around a give point, when the values of r , ϑ and φ are changed. To calculate the tangent vector to a r -coordinate line we consider

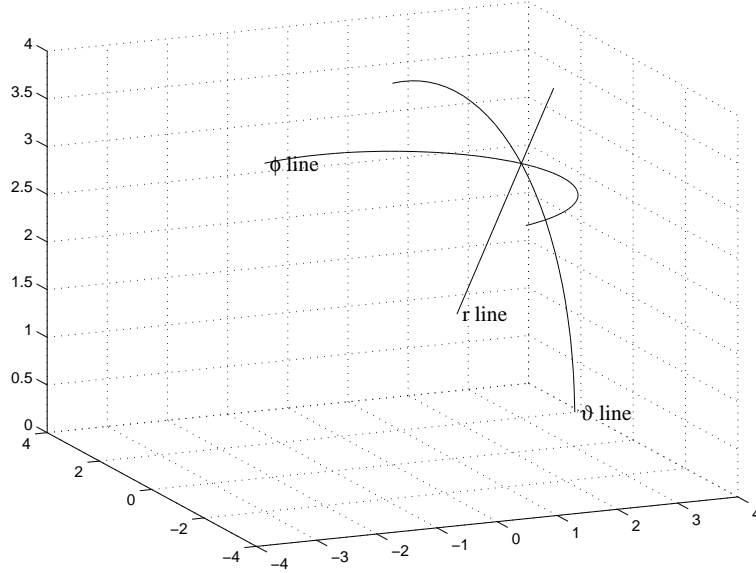


Figure 3.7: Coordinate lines of the spherical coordinate system

ϑ and φ to be constant and we differentiate equation 3.9 with respect to r

$$\frac{d\vec{r}}{dr} = \begin{pmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{pmatrix}$$

For the length of this vector we find

$$\left| \frac{d\vec{r}}{dr} \right| = \sqrt{\sin^2(\vartheta)[\cos^2(\varphi) + \sin^2(\varphi)] + \cos^2(\vartheta)} = 1$$

so that the tangent vector is already a unit vector and we are able to define the unit vector \vec{u}_r of the spherical coordinate system

$$\vec{u}_r = \begin{pmatrix} \sin(\vartheta) \cos(\varphi) \\ \sin(\vartheta) \sin(\varphi) \\ \cos(\vartheta) \end{pmatrix} = \vec{u}_x \sin(\vartheta) \cos(\varphi) + \vec{u}_y \sin(\vartheta) \sin(\varphi) + \vec{u}_z \cos(\vartheta) \quad (3.10)$$

In the next step we calculate the tangent vector to a ϑ -coordinate line. That means that we have to differentiate equation 3.9 with respect to ϑ

$$\frac{d\vec{r}}{d\vartheta} = \begin{pmatrix} r \cos(\vartheta) \cos(\varphi) \\ r \cos(\vartheta) \sin(\varphi) \\ -r \sin(\vartheta) \end{pmatrix}$$

for the norm of this vector we get

$$\left| \frac{d\vec{r}}{d\vartheta} \right| = \sqrt{r^2 \cos^2(\vartheta)[\cos^2(\varphi) + \sin^2(\varphi)] + r^2 \sin^2(\vartheta)} = r$$

Now we are ready to define the unit vector \vec{u}_ϑ in ϑ -direction

$$\vec{u}_\vartheta = \frac{\frac{d\vec{r}}{d\vartheta}}{\left|\frac{d\vec{r}}{d\vartheta}\right|} = \begin{pmatrix} \cos(\vartheta) \cos(\varphi) \\ \cos(\vartheta) \sin(\varphi) \\ -\sin(\vartheta) \end{pmatrix} = \vec{u}_x \cos(\vartheta) \cos(\varphi) + \vec{u}_y \cos(\vartheta) \sin(\varphi) - \vec{u}_z \sin(\vartheta)$$

To get the tangent vectors to the φ -coordinate lines we differentiate equation 3.9 with respect to φ

$$\frac{d\vec{r}}{d\varphi} = \begin{pmatrix} -r \sin(\vartheta) \sin(\varphi) \\ r \sin(\vartheta) \cos(\varphi) \\ 0 \end{pmatrix}$$

and for its norm we get

$$\left|\frac{d\vec{r}}{d\varphi}\right| = \sqrt{r^2 \sin^2(\vartheta) [\sin^2(\varphi) + \cos^2(\varphi)]} = r \sin(\vartheta)$$

Hence we get for the unit vector in φ -direction

$$\vec{u}_\varphi = \frac{\frac{d\vec{r}}{d\varphi}}{\left|\frac{d\vec{r}}{d\varphi}\right|} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} = -\vec{u}_x \sin(\varphi) + \vec{u}_y \cos(\varphi)$$

which is identical with the unit vector \vec{u}_φ of the cylindrical coordinate system, already defined in equation 3.8.

3.4.3 Orthogonal coordinates

We consider an arbitrary orthogonal coordinate system which is defined by the following mapping

$$(u_1, u_2, u_3) \mapsto \vec{r} = \begin{pmatrix} x(u_1, u_2, u_3) \\ y(u_1, u_2, u_3) \\ z(u_1, u_2, u_3) \end{pmatrix}$$

The tangent vector \vec{T}_i to the coordinate line i is then given by

$$\vec{T}_i = \frac{\partial \vec{r}}{\partial u_i} = \begin{pmatrix} \frac{\partial x}{\partial u_i} \\ \frac{\partial y}{\partial u_i} \\ \frac{\partial z}{\partial u_i} \end{pmatrix}$$

Of course this tangent vector will normally not have a unit length, so we have to define the unit vector to the coordinate line of u_i by

$$\vec{u}_i = \frac{\vec{T}_i}{|\vec{T}_i|} = \frac{1}{h_i} \frac{\partial \vec{r}}{\partial u_i} \quad \text{with} \quad h_i = |\vec{T}_i|$$

With the help of the last equation we find

$$\frac{d\vec{r}}{du_i} = \vec{u}_i(u_1, u_2, u_3) h_i(u_1, u_2, u_3)$$

Hence we find for the total differential of the incremental way element

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3 = h_1 \vec{u}_1 du_1 + h_2 \vec{u}_2 du_2 + h_3 \vec{u}_3 du_3$$

The values h_i that give by a multiplication with du_i the metrical length in the 3-dimensional space are called **metrical factors**.

3.4.4 Nabla operator of an arbitrary orthogonal coordinate system

If we consider a real valued function in an arbitrary orthogonal coordinate system this reads

$$\phi = \phi(u_1, u_2, u_3)$$

For the total differential of this function we find

$$d\phi = \frac{\partial\phi}{\partial u_1} du_1 + \frac{\partial\phi}{\partial u_2} du_2 + \frac{\partial\phi}{\partial u_3} du_3$$

or with the help of the metrical factors

$$d\phi = \frac{\partial\phi}{h_1 \partial u_1} h_1 du_1 + \frac{\partial\phi}{h_2 \partial u_2} h_2 du_2 + \frac{\partial\phi}{h_3 \partial u_3} h_3 du_3$$

This may be expressed by the dot product of the gradient of the function ϕ and the incremental way element $d\vec{r}$ since the unit vectors \vec{u}_i of the coordinate system are mutual orthogonal

$$d\phi = \left(\vec{u}_1 \frac{\partial\phi}{h_1 \partial u_1} + \vec{u}_2 \frac{\partial\phi}{h_2 \partial u_2} + \vec{u}_3 \frac{\partial\phi}{h_3 \partial u_3} \right) \cdot (\vec{u}_1 h_1 du_1 + \vec{u}_2 h_2 du_2 + \vec{u}_3 h_3 du_3)$$

Hence the nabla operator of an arbitrary orthogonal coordinate system reads

$$\vec{\nabla} = \left(\vec{u}_1 \frac{\partial}{h_1 \partial u_1} + \vec{u}_2 \frac{\partial}{h_2 \partial u_2} + \vec{u}_3 \frac{\partial}{h_3 \partial u_3} \right)$$

Example Nabla operator of the cylindric coordinate system

The cylindric coordinate system is defined by equation 3.5. Hence we find for the metrical factors h_i

$$\begin{aligned} h_1 &= \left| \frac{\partial \vec{r}}{\partial \rho} \right| = 1 \\ h_2 &= \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \rho \\ h_3 &= \left| \frac{\partial \vec{r}}{\partial z} \right| = 1 \end{aligned}$$

thus the nabla operator of the cylindric coordinate system is given by

$$\vec{\nabla} = \left(\vec{u}_\rho \frac{\partial}{\partial \rho} + \vec{u}_\phi \frac{\partial}{\rho \partial \phi} + \vec{u}_z \frac{\partial}{\partial z} \right)$$

Example Nabla operator of the spherical coordinate system

The spherical coordinate system is defined by equation 3.9. Hence we find for the metrical factors h_i

$$\begin{aligned} h_1 &= \left| \frac{\partial \vec{r}}{\partial r} \right| = 1 \\ h_2 &= \left| \frac{\partial \vec{r}}{\partial \vartheta} \right| = r \\ h_3 &= \left| \frac{\partial \vec{r}}{\partial \phi} \right| = r \sin(\vartheta) \end{aligned}$$

Hence we find for the nabla operator of the spherical coordinate system

$$\vec{\nabla} = \left(\vec{u}_r \frac{\partial}{\partial r} + \vec{u}_\vartheta \frac{\partial}{r \partial \vartheta} + \vec{u}_\phi \frac{\partial}{r \sin(\vartheta) \partial \phi} \right)$$

3.5 Surfaces

We already used one kind of a surface, namely the graph of a real valued function defined on a plane

$$\text{graph } f = \{x, y, f(x, y) \in \mathbf{R}^3 \mid x, y \in U \subset \mathbf{R}^2\}$$

But there exist surfaces that can not be graphs of a function since in this case for each element $x, y \in \mathbf{R}^2$ there must be only one z with $(x, y, z) \in S$. Instead we will consider a surface to be a mapping from a 2-dimensional space into the 3-dimensional space

$$(u, v) \in \mathbf{R}^2 \mapsto \vec{S}(u, v) \in \mathbf{R}^3$$

Definition A parametrized surface is a function $\vec{S} : D \subset \mathbf{R}^2 \mapsto \mathbf{R}^3$, where D is some domain in \mathbf{R}^2 . The surface \vec{S} is the image of the mapped domain

$$(u, v) \in D \subset \mathbf{R}^2 \mapsto \vec{S}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$$

If \vec{S} is differentiable or is of class \mathbf{C}^1 , which is the same as saying that $x(u, v), y(u, v)$

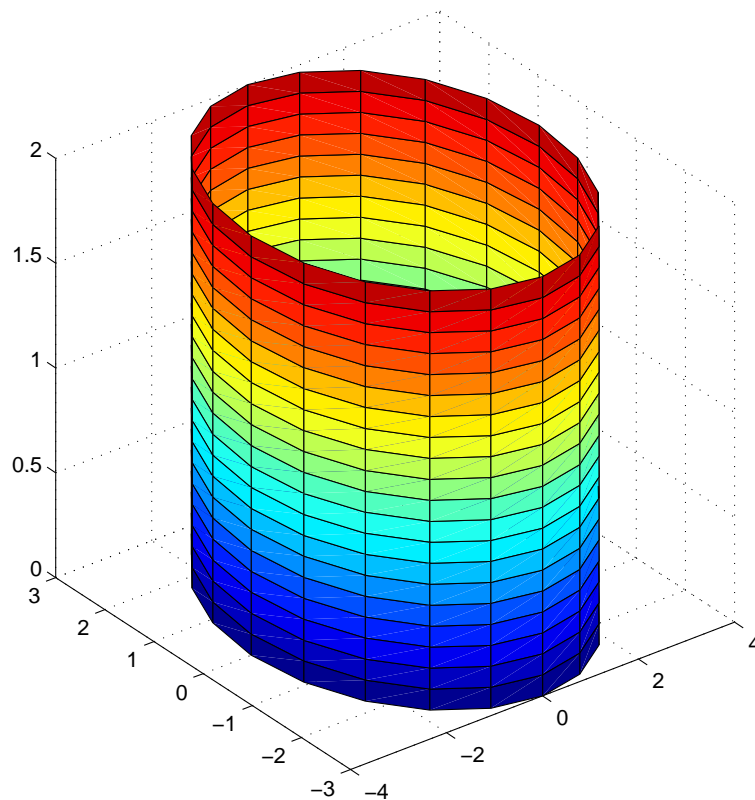


Figure 3.8: Surface of a cylinder

and $z(u, v)$ are differentiable or \mathbf{C}^1 functions of (u, v) , we call \vec{S} a differentiable or a \mathbf{C}^1

surface.

Example Surface of a cylinder

We will consider the following mapping

$$\{0 \leq u < 2\pi, 0 \leq v \leq h\} \mapsto \vec{S} = \begin{pmatrix} \rho \cos(u) \\ \rho \sin(u) \\ v \end{pmatrix} \quad (3.11)$$

Figure 3.8 shows the result of the mapping in a 3-dimensional coordinate system, while the following lines give the MATLAB-code to produce this picture

```
clear;  
rho = 3;  
[U,V] = meshgrid(0:pi/10:2*pi,0:0.1:2);  
X = rho.*cos(U);  
Y = rho.*sin(U);  
Z = V;  
surf(X,Y,Z);  
axis square;
```

Example Surface of a torus

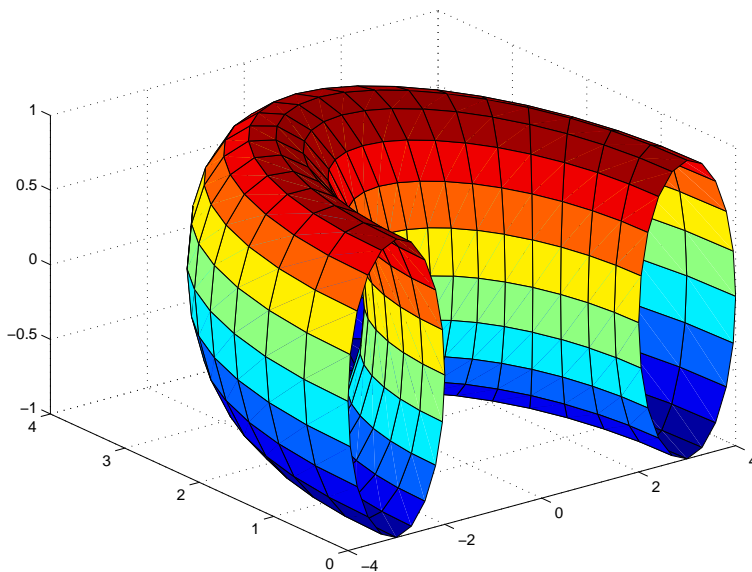


Figure 3.9: Surface of a torus

As a second example will consider a mapping, that is only given in MATLAB notation and which produces the picture shown in Figure 3.9

```
clear;
```

```

rho = 3.0;
[U,V] = meshgrid(0:pi/20:pi,0:pi/10:2*pi);
X      = (rho + cos(V)).*cos(U);
Y      = (rho + cos(V)).*sin(U);
Z      = sin(V);
surf(X,Y,Z)

```

3.5.1 Tangent vectors to the surface

If for example we fix one of the two coordinates u or v we get a coordinate line on the surface. We can of course calculate the tangent vector to this curve on the surfaces for example in a given point u_0, v_0

$$\vec{T}_u = \frac{\partial \vec{S}}{\partial u} \Big|_{(u_0, v_0)} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial u} \\ \frac{\partial y(u,v)}{\partial u} \\ \frac{\partial z(u,v)}{\partial u} \end{pmatrix} \Big|_{(u_0, v_0)} \quad \vec{T}_v = \frac{\partial \vec{S}}{\partial v} \Big|_{(u_0, v_0)} = \begin{pmatrix} \frac{\partial x(u,v)}{\partial v} \\ \frac{\partial y(u,v)}{\partial v} \\ \frac{\partial z(u,v)}{\partial v} \end{pmatrix} \Big|_{(u_0, v_0)}$$

Definition: We say that the surface \vec{S} is smooth at $\vec{S}(u, v)$ if the two tangent vectors \vec{T}_u and \vec{T}_v exist and if $\vec{T}_u \times \vec{T}_v \neq 0$. That means a normal vector to the surface at the point (u, v) can be defined by

$$\vec{N} = \vec{T}_u \times \vec{T}_v \quad \text{or as unit vector} \quad \vec{u}_N = \frac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|}$$

3.5.2 Area of a surfaces

To calculate the total area of a surface we first consider a very small area in the definition domain near the point (u, v) which has the extentions du and dv and how this area is mapped onto the surface in the 3-dimensional space. This process is illustrated in Figure 3.10. The small parallelogram dA shown on the surface may be calculated with

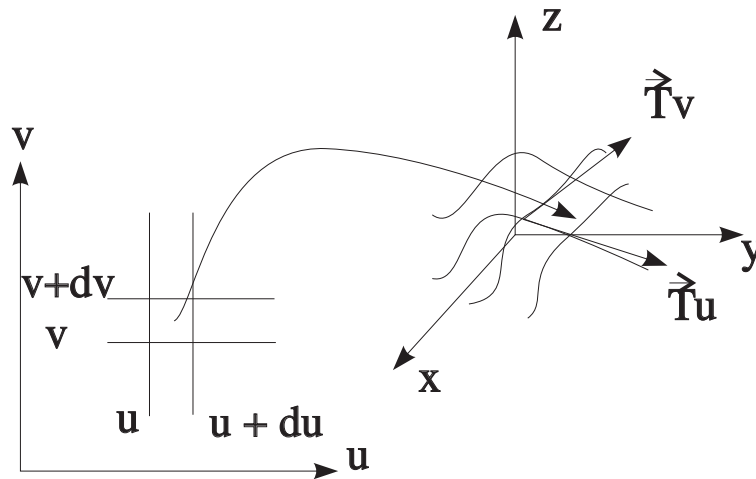


Figure 3.10: Mapping of a domain element

the help of the tangent vectors \vec{T}_u and \vec{T}_v to the coordinate lines in the mapped point $\vec{S}(u, v)$. For the length of each side dS_u and dS_v we have

$$dS_u = |\vec{T}_u|du \quad dS_v = |\vec{T}_v|dv$$

and hence the value of the small area dA is given by

$$dA = dS_u dS_v \sin(\varphi_{uv}) = |\vec{T}_u \times \vec{T}_v| dudv$$

where the last expression reflects the fact that the cross product of two vectors is proportional to the sine of the angle φ_{uv} between them. With the help of this considerations we are now able to define how to calculate the total area of a parametrized surface.

Definition: We consider a parametrized surface $\vec{S} : (u, v) \in D \subset \mathbf{R}^2 \mapsto \mathbf{R}^3$ then the total area A of the surface may be calculated by

$$A = \iint_D |\vec{T}_u \times \vec{T}_v| dudv$$

If the total surface is a union of smaller surfaces its area is equal to the sum of these.

Example: Compute the surface of a cylinder given by the mapping 3.11.

For the tangent vectors \vec{T}_u and \vec{T}_v we find

$$\vec{T}_u = \frac{\partial \vec{S}}{\partial u} = \rho \begin{pmatrix} -\sin(u) \\ \cos(u) \\ 0 \end{pmatrix} = \rho \vec{u}_\varphi \quad \vec{T}_v = \frac{\partial \vec{S}}{\partial v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{u}_z$$

Hence we have to solve the following integration

$$A = \int_0^{2\pi} \int_0^h \rho |\vec{u}_\varphi \times \vec{u}_z| dudv = \rho \int_0^{2\pi} du \int_0^h dv = 2\pi\rho h$$

and end up with a well known result.

Chapter 4

Integrals over Paths and Surfaces

4.1 The path integral

This section introduces the concept of the path integral, this is one of several ways in which integrals of functions of one variable can be generalized to functions of several variables. We will first define a path integral over a real valued function.

Definition: The path integral, or the integral of $f(x, y, z)$ along the path $\vec{\sigma}$ is defined when $\vec{\sigma} : t \in [a, b] \mapsto \mathbf{R}^3$ is of class \mathbf{C}^1 and when the composite function $t \mapsto f(x(t), y(t), z(t))$ is continuous on $[a, b]$. We define the path integral by

$$\int_s f ds = \int_a^b f(x(t), y(t), z(t)) \left| \frac{d\vec{\sigma}}{dt} \right| dt$$

Sometimes $\int_s f ds$ is also written as

$$\int_s f(x, y, z) ds \quad \text{or} \quad \int_s f(\vec{\sigma}) ds$$

Example: Let $\vec{\sigma}$ be a helix defined by $t \in [0, 2\pi] \mapsto \vec{\sigma}(t) = (\cos(t), \sin(t), t)$ and let $f(x, y, z)$ be $x^2 + y^2 + z^2$. Evaluate the integral $\int_s f(x, y, z) ds$.

For the tangent vector and its absolute value we get

$$\frac{d\vec{\sigma}}{dt} = \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 1 \end{pmatrix} \rightarrow \left| \frac{d\vec{\sigma}}{dt} \right| = \sqrt{2}$$

and the result for the composite function reads

$$f(x(t), y(t), z(t)) = \cos^2(t) + \sin^2(t) + t^2 = 1 + t^2$$

Hence we have to solve the following integral

$$\int_s f(x, y, z) ds = \int_0^{2\pi} (1 + t^2) \sqrt{2} dt = \sqrt{2} \left[t + \frac{t^3}{3} \right]_0^{2\pi} = \frac{\sqrt{2} 2\pi}{3} [3 + 4\pi^2]$$

The path integral over a real valued function does not depend on the orientation of the curve, since the orientation of the path is lost during the evaluation the infinitesimal way element ds by $|d\vec{\sigma}/dt| dt$.

4.1.1 Potential function of charge distribution along a filament

Suppose a curve $\vec{r}_q(s)$ is given in natural parametrization. Along this filament a one dimensional charge distribution $\lambda(s) = dQ/ds$ is defined. How to calculate the potential function of this charge distribution at a given point \vec{r}_p ? For purpose of illustration Figure 4.1 shows the considered configuration. The influence of the charge dQ located

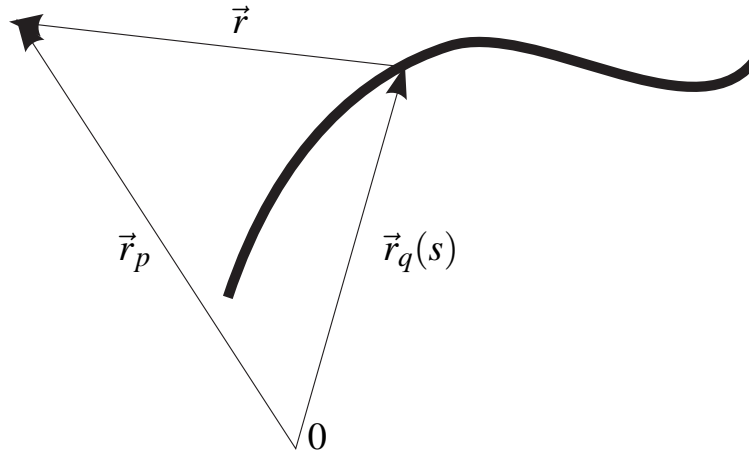


Figure 4.1: Potential function of an charge distribution a long a filament

at the source point $\vec{r}_q(s)$ onto the point \vec{r}_p is given by

$$d\phi = \frac{1}{4\pi\epsilon_0} \frac{dQ}{r}$$

where r denotes the absolute value of the vector $\vec{r} = \vec{r}_p - \vec{r}_q(s)$. Considering the function of the charge distribution $\lambda(s)$ we find for the total potential function ϕ in the point \vec{r}_p

$$\phi(\vec{r}_p) = \frac{1}{4\pi\epsilon_0} \int_s \frac{\lambda(s)ds}{|\vec{r}_p - \vec{r}_q(s)|}$$

Example: Given a constant charge distribution λ_0 on a circle with radius ρ_0 in the xy -plane. Calculate the potential function along the z -axes.

The easiest way to describe a circle in the xy -plane is to use cylindrical coordinates. Hence we get for the vectors \vec{r}_q and \vec{r}_p

$$\vec{r}_q = \rho_0 \vec{u}_\rho \quad \vec{r}_p = z \vec{u}_z$$

The distance between the source point \vec{r}_q and the point \vec{r}_p the potential function has to be evaluated reads

$$r = |\vec{r}_p - \vec{r}_q| = \sqrt{z^2 + \rho^2}$$

To describe the circle we have to variate the angle φ between 0 and 2π , while the incremental way element ds equals $\rho_0 d\varphi$. To calculate the potential function along the z -axes we have to solve the following integral

$$\phi(z) = \frac{1}{4\pi\epsilon_0} \int_0^{2\pi} \frac{\lambda_0 \rho_0}{\sqrt{z^2 + \rho_0^2}} d\varphi = \frac{1}{4\pi\epsilon_0} \frac{2\pi \rho_0 \lambda_0}{\sqrt{z^2 + \rho_0^2}} = \frac{1}{4\pi\epsilon_0} \frac{2\pi \rho_0 \lambda_0}{\rho_0} \frac{1}{\sqrt{1 + \left(\frac{z}{\rho_0}\right)^2}}$$

In the last equation the term $2\pi\rho_0\lambda_0$ is equal to the total charge Q distributed along the circle. So the first two terms represent the potential ϕ_0 a charge Q induces at a distance ρ_0 .

4.1.2 Static electric field of a charge distribution along a filament

As in the last section we suppose a filament is defined by a path $\vec{r}_q(s)$ with natural parametrization. On the path a one dimensional charge distribution $\lambda(s) = dQ/ds$ is defined. How to calculate the static electric field in a given point \vec{r}_p ? We start by considering an infinitesimal small charge dQ located at a source point $\vec{r}_q(s)$ using the notation of Figure 4.1. According to Coulombs law [5] in vector formulation this charge will contribute the following infinitesimal electric field $d\vec{E}$ to the total field

$$d\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\vec{r}}{r^3} dQ$$

To get the total electric field in the considered point \vec{r}_p we have to sum up all contributions of the charges along the filament. This leads to the following integral

$$\vec{E}(\vec{r}_p) = \frac{1}{4\pi\epsilon_0} \int_s \frac{\vec{r}(s)}{r^3(s)} \lambda(s) ds = \frac{1}{4\pi\epsilon_0} \int_s \frac{\vec{r}_p - \vec{r}_q(s)}{|\vec{r}_p - \vec{r}_q(s)|^3} \lambda(s) ds$$

4.1.3 The line integral

In the last section the orientation of the curve did not contribute to the result of the integration. That is not the case for the line integral we will define now. Consider a small charge q is moved along an orientated path $\vec{\sigma}$ in an arbitrary static electric field as illustrated in Figure 4.2. In each point of the path the charge will experience a force

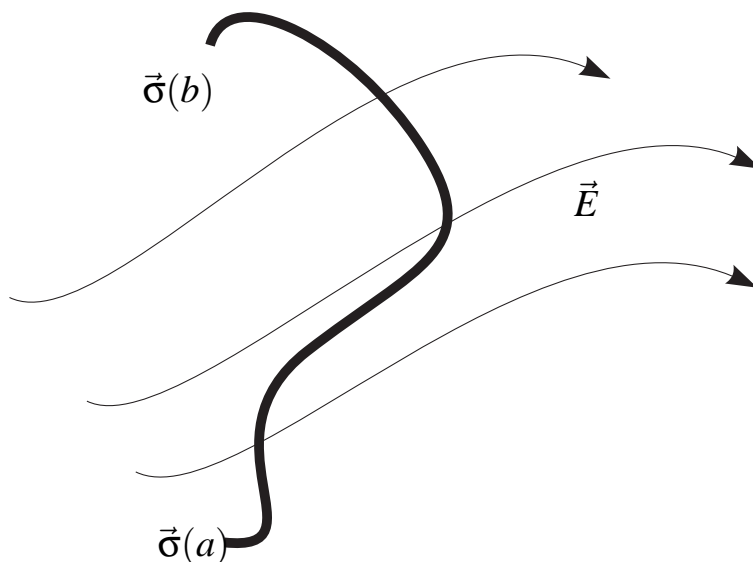


Figure 4.2: Moving a small charge in a static electric field

$\vec{F} = q\vec{E}(s)$ depending on the electric field at that point. If we move the charge an

infinitesimal small way $d\vec{\sigma}$ along the path a work dW will have to be done, that is equal to the dot product between the force \vec{F} and the way element, so we have

$$dW = \vec{F}(s) \cdot d\vec{\sigma} = q\vec{E}(s) \cdot d\vec{\sigma}$$

To get the total work we have to sum up all contributions along the orientated path $\vec{\sigma}(s)$, which leads to the following integrals

$$W = \int_s \vec{F}(s) \cdot d\vec{\sigma} = q \int_s \vec{E}(s) \cdot d\vec{\sigma}$$

Now we are ready to formulate the definition of the line integral.

Definition: Let \vec{F} be a vector field defined on \mathbf{R}^3 which is continuous on the \mathbf{C}^1 path $\vec{\sigma} : [a, b] \mapsto \mathbf{R}^3$. We define $\int_s \vec{F} \cdot d\vec{\sigma}$, the line integral of \vec{F} along the path $\vec{\sigma}$, by the following formula

$$\int_s \vec{F} \cdot d\vec{\sigma} = \int_a^b \vec{F}(\vec{\sigma}(t)) \cdot \frac{d\vec{\sigma}}{dt} dt$$

This line integral is independent of the chosen way as long as the vector field is conservative, i.e. as long as \vec{F} is given by the gradient of a real-valued function.

Example: Consider a homogenous static electric field $\vec{E} = E_0\vec{u}_x$ and two paths in the xy-plane. One path along the x-axis from the origin to $x = 2\rho$ and the another along a half circle lying in forth quadrant, starting in the origin and ending at the point $x = 2\rho$ also. Calculate the line integral $\int_s \vec{E} d\vec{\sigma}$ along the two different paths.

We start our calculation by giving a parametrisation of the first path

$$t \in [0, 2\rho] \mapsto \vec{\sigma}(t) = \vec{u}_x t$$

so we have $\frac{d\vec{\sigma}}{dt} = \vec{u}_x$ and we have to solve the following integration

$$\int_0^{2\rho} \vec{E} \cdot d\vec{\sigma} = \int_0^{2\rho} E_0\vec{u}_x \cdot \vec{u}_x dt = E_0 2\rho$$

Two parametrize the second path, we use the following mapping

$$t \in [0, \pi] \mapsto \vec{\sigma}(t) = \rho\vec{u}_x - \rho\vec{u}_x \cos(t) - \rho\vec{u}_y \sin(t)$$

For the differential way element $d\vec{\sigma}$ we find

$$d\vec{\sigma} = \rho [\sin(t)\vec{u}_x - \cos(t)\vec{u}_y] dt$$

so we have to solve the following integration

$$\int_s \vec{E} d\vec{\sigma} = \int_0^\pi E_0\vec{u}_x \cdot \rho [\sin(t)\vec{u}_x - \cos(t)\vec{u}_y] dt = \int_0^\pi E_0\rho \sin(t) dt = E_0 2\rho$$

Thus we see, that we get the same result independent of the path we used to do the line integration. We will see later on that this is true for arbitrary paths between to different points in a static electric field, since a static electric field can allways be described as gradient of the potential function.

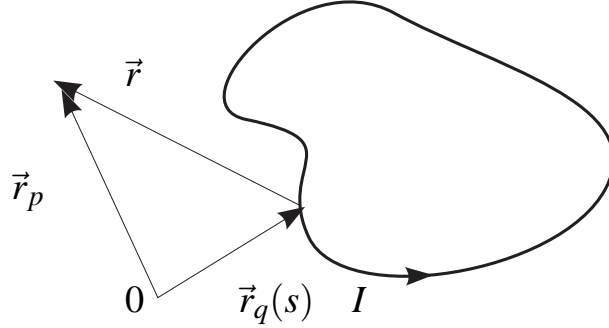


Figure 4.3: Static magnetic field

4.1.4 Static magnetic field due to a current filament

In section 4.1.2 we have already discussed an integration along a path which gave a vector as result. The same holds true for the law of Biot and Savart ??, which tells us how to calculate the static magnetic field due to a current following in an arbitrary shaped wire. The geometry and direction of the flowing current is described by a natural parametrized curve $\vec{r}_q(s)$ with its infinitesimal way element $d\vec{r}_q$. The point where the static field is going to be calculated is denoted by \vec{r}_p resulting in a difference vector $\vec{r} = \vec{r}_p - \vec{r}_q$ between the point \vec{r}_p and the source of the magnetic field. With these definitions the law of Biot-Savart reads

$$\vec{H}(\vec{r}_p) = \frac{I}{4\pi} \oint_C d\vec{r}_q \times \frac{\vec{r}}{r^3} = \frac{I}{4\pi} \oint_C d\vec{r}_q \times \frac{\vec{r}_p - \vec{r}_q}{|\vec{r}_p - \vec{r}_q|^3}$$

To illustrate how this formulare works we will consider the following example.

Example: A current I is flowing in a circle shaped wire in the xy -plane with its center at the origin. Calculate the static magnetic field along the z -axis.

To represent the points along the z -axes we define $\vec{r}_p = z\vec{u}_z$ while the circle in the plane may be described by the following mapping

$$\varphi \in [0, 2\pi] \mapsto \vec{r}_q(\varphi) = \rho [\cos(\varphi)\vec{u}_x + \sin(\varphi)\vec{u}_y]$$

so we find for the infinitesimal way element along the path $d\vec{r}_q$

$$\frac{d\vec{r}_q}{d\varphi} = \rho \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix} = \rho\vec{u}_\varphi \quad \rightarrow \quad d\vec{r}_q = \rho\vec{u}_\varphi d\varphi$$

For the vector \vec{r} we have

$$\vec{r} = z\vec{u}_z - \rho\vec{u}_\rho \quad \rightarrow \quad r = |\vec{r}| = \sqrt{z^2 + \rho^2}$$

Before we can evaluate the integral we first examine the cross product

$$d\vec{r}_q \times \vec{r} = \rho d\varphi \vec{u}_\varphi \times (z\vec{u}_z - \rho\vec{u}_\rho) = \rho(z\vec{u}_\rho + \rho\vec{u}_z)$$

hence we have to solve the following integration

$$\vec{H}(z) = \frac{I}{4\pi} \rho \int_0^{2\pi} \frac{z\vec{u}_\rho + \rho\vec{u}_z}{(\sqrt{z^2 + \rho^2})^3} d\varphi$$

This integral may be splitted into two ones, in the first the unit vector \vec{u}_ρ has to be integrated from $\varphi = 0$ to $\varphi = 2\pi$, which gives zero as its result. In the second term there is no φ dependence, so 2π will be its result. Hence this leads to the following equation for the static magnetic field along the z-axis.

$$\vec{H}(z) = \frac{I}{2\pi} \frac{\rho^2}{(\sqrt{z^2 + \rho^2})^3} \vec{u}_z$$

4.2 The surface integral

In section 3.5.2 we already discussed how to calculate the area of a parametrized surface. Now we will go on and examine how to integrate a scalar function which is defined on the surface.

4.2.1 Surface integral of a real valued function

Consider the following situation, given an arbitrary surface $\vec{S}(u, v)$. On this surface we have a given charge distribution $\sigma(u, v)$. What is the total charge Q located on the surface? To solve this problem we first consider an infinitesimal area dA of the surface at a given point (u, v) . This area will carry the charge $dQ = \sigma(u, v)dA$. The total charge is the sum of this infinitesimal charges.

$$Q = \iint_S \sigma dA$$

To explicitly carry out this integration, we remember how we have calculated the area of a parametrized surface. The following definition will help to solve the problem.

Definition: If f is a real valued continuous function defined on \vec{S} , we define the integral of f over S to be

$$\iint_S f dA = \iint_D f(\vec{S}(u, v)) |\vec{T}_u \times \vec{T}_v| du dv$$

Example : We consider the following charge distribution

$$\sigma(x, y) = \sigma_0 \left(\sqrt{\left(\frac{x}{\rho_0}\right)^2 + \left(\frac{y}{\rho_0}\right)^2} \right)^n$$

on a circular disc with radius ρ_0 . Calculate the total charge on the disc.

To describe the circular disc, we introduce the following parametrization.

$$(\rho, \varphi) \in [(0, \rho_0) (0, 2\pi)] \mapsto \vec{S}(\rho, \varphi) = \rho \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}$$

So we get for the tangent vectors of the ρ and φ coordinate lines

$$\vec{T}_\rho = \frac{d\vec{S}}{d\rho} = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix} = \vec{u}_\rho \quad \vec{T}_\varphi = \frac{d\vec{S}}{d\varphi} = \rho \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix} = \rho \vec{u}_\varphi$$

and hence for the incremental area element dA we find

$$d\vec{A} = (\vec{T}_\rho \times \vec{T}_\varphi) d\rho d\varphi = \rho \vec{u}_z d\rho d\varphi \quad \rightarrow \quad dA = \rho d\rho d\varphi$$

With the introduced mapping the function $\sigma(x, y)$ becomes a function of (ρ, φ)

$$\sigma(\rho, \varphi) = \sigma_0 \left(\sqrt{\left(\frac{\rho \cos(\varphi)}{\rho_0}\right)^2 + \left(\frac{\rho \sin(\varphi)}{\rho_0}\right)^2} \right)^n = \sigma_0 \left(\frac{\rho}{\rho_0}\right)^n$$

In order to calculate the total charge of the disk we have to perform the following integration:

$$Q = \int_0^{2\pi} \int_0^{\rho_0} \sigma_0 \left(\frac{\rho}{\rho_0}\right)^n \rho d\rho d\varphi = 2\pi\sigma_0 \frac{\rho_0^2}{n+2}$$

For a constant charge distribution ($n = 0$) we get the known result $Q = \pi\sigma_0\rho_0^2$ from the last equation.

4.2.2 Surface integral of a vector valued function

To give an example for this integration we have a look at Figure 4.4. It shows the

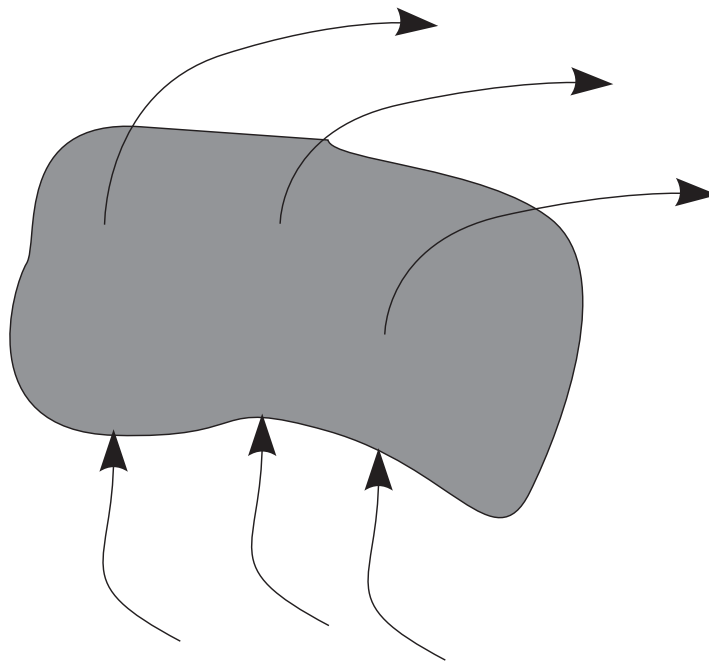


Figure 4.4: Total current through an arbitrary area

field lines of a current density field $\vec{J}(x, y, z)$ flowing through the shown area. How to calculate the total current flowing through the surface. If we consider this problem we must remember, that of course the flow lines are not always perpendicular to the surface they pass. Considering a small area $d\vec{A}$ the net current dI through this area is given by

$$dI = \vec{J} \cdot d\vec{A} = |\vec{J}| |d\vec{A}| \cos(\varphi_{JA})$$

Here the dot product of the vectors assures that only the net flux through the area is counted. To calculate the total current we have to sum up all current elements dI . This leads us to the following definition.

Definition: Let \vec{F} be a vector field defined on the surface $\vec{S}(u, v)$. The surface integral of \vec{F} over $\vec{S}(u, v)$ is defined by

$$\int \int_S \vec{F} \cdot d\vec{A} = \int \int_D \vec{F}(\vec{S}(u, v)) \cdot (\vec{T}_u \times \vec{T}_v) du dv$$

Example: Consider the static electric field of a charge Q located in the origin of a coordinate system. Calculate the electric flux thru a cylinder with height h and radius ρ_0 surrounding symmetrical the charge.

We start our considerations by remembering the electric field \vec{E} and the electric displacement field \vec{D} of the charge.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \vec{u}_r \quad \vec{D} = \epsilon_0 \vec{E} = \frac{Q}{4\pi} \frac{1}{r^2} \vec{u}_r$$

Now we will give a mapping that describes the cover of the cylinder

$$(\varphi, z) \in \left([0, 2\pi] \left[-\frac{h}{2}, \frac{h}{2} \right] \right) \mapsto \vec{S}(\varphi, z) = \begin{pmatrix} \rho_0 \cos(\varphi) \\ \rho_0 \sin(\varphi) \\ z \end{pmatrix}$$

On the cover of the cylinder the electric displacement vector is given by

$$\vec{D} = \frac{Q}{4\pi} \frac{1}{\rho_0^2 + z^2} \vec{u}_r$$

To calculate $d\vec{A}$ of the cylinder cover, we first have to calculate the tangent vectors to the coordinate lines

$$\vec{T}_\varphi = \frac{d\vec{S}}{d\varphi} = \begin{pmatrix} -\rho_0 \sin(\varphi) \\ \rho_0 \cos(\varphi) \\ 0 \end{pmatrix} = \rho_0 \vec{u}_\varphi \quad \vec{T}_z = \frac{d\vec{S}}{dz} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \vec{u}_z$$

so we find for the incremental area element $d\vec{A}$ of the cylinder cover

$$d\vec{A} = (\vec{T}_\varphi \times \vec{T}_z) d\varphi dz = \rho_0 \vec{u}_\rho d\varphi dz$$

Thus the integral over the cover reads

$$\int \int_C \vec{D} \cdot d\vec{A} = \frac{Q}{4\pi} \int_0^{2\pi} \int_{-h/2}^{h/2} \frac{\rho_0}{\rho_0^2 + z^2} \vec{u}_r \cdot \vec{u}_\rho dz d\varphi$$

Before we can solve the above integrals we first have to evaluate the dot product between the unit vector \vec{u}_r of the spherical and \vec{u}_ρ of the cylindrical coordinate system. Their representation in cartesian coordinates are given in equation 3.10 and 3.7. So the result of the dot product is

$$\vec{u}_r \cdot \vec{u}_\rho = \sin(\vartheta) \quad \text{with} \quad \sin(\vartheta) = \frac{\rho_0}{\sqrt{\rho_0^2 + z^2}} \quad \text{on the cylinder cover}$$

So the following integrals have to be solved

$$\int \int_C \vec{D} \cdot d\vec{A} = \frac{Q}{4\pi} \int_0^{2\pi} d\varphi \int_{-h/2}^{h/2} \frac{\rho_0^2}{(\rho_0^2 + z^2)^{3/2}} dz$$

As result we find

$$\iint_C \vec{D} \cdot d\vec{A} = Q \frac{h/2}{\sqrt{(h/2)^2 + \rho_0^2}}$$

To go on with our example we now have to consider the electric flux going thru the top of the cylinder. A mapping describing this surface reads

$$(\rho, \varphi) \in ([0, \rho_0] [0, 2\pi]) \mapsto \vec{S}(\rho, \varphi) = \begin{pmatrix} \rho \cos(\varphi) \\ \rho \sin(\varphi) \\ h/2 \end{pmatrix}$$

The electric displacement vector on this surface is given by

$$\vec{D} = \frac{Q}{4\pi} \frac{1}{\rho^2 + (h/2)^2} \vec{u}_r$$

For the tangent vectors to the coordinate lines we find

$$\vec{T}_\rho = \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} = \vec{u}_\rho \quad \vec{T}_\varphi = \rho \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} = \rho \vec{u}_\varphi$$

and therefore the incremental area element is given by $d\vec{A} = \rho \vec{u}_z d\rho d\varphi$ and we have to solve the following integrals

$$\iint_T \vec{D} \cdot d\vec{A} = \frac{Q}{4\pi} \int_0^{2\pi} \int_0^{\rho_0} \frac{\rho}{\rho^2 + (h/2)^2} \vec{u}_r \cdot \vec{u}_z d\rho d\varphi$$

Before one can solve the above integrals one again first has to calculate the dot product of the unit vectors which gives the following result

$$\vec{u}_r \cdot \vec{u}_z = \cos(\vartheta) \quad \text{with} \quad \cos(\vartheta) = \frac{h/2}{\sqrt{\rho^2 + (h/2)^2}} \quad \text{on top of the cylinder}$$

Doing the integration we end up with following result giving the electric flux thru the top of the cylinder surface which for reason of symmetry is equal to the flux thru the bottom of the cylinder

$$\iint_T \vec{D} \cdot d\vec{A} = \frac{Q}{2} \left[1 - \frac{h/2}{\sqrt{\rho_0^2 + (h/2)^2}} \right]$$

To get the total electric flux thru the cylinder we have to add up the fluxes thru the cover, the top and the bottom of the cylinder, which yields

$$\iint_S \vec{D} \cdot d\vec{A} = \iint_C \vec{D} \cdot d\vec{A} + 2 \iint_T \vec{D} \cdot d\vec{A} = Q$$

So we proofed in this spezial case Gauss law, which states that the total electric flux flowing out of a closed surface is equal to the charges enclosed.

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